Monte Carlo Prediction & Control

CMPUT 261: Introduction to Artificial Intelligence

S&B §5.0-5.5, 5.7

Lecture Outline

1. Recap & Logistics
2. Monte Carlo Prediction
3. Estimating Action Values
4. Monte Carlo Control
5. Importance Sampling
6. Off-Policy Monte Carlo Control

After this lecture, you should be able to:

- explain how Monte Carlo estimation for state values works
- trace an execution of first-visit Monte Carlo Prediction
- explain the difference between prediction and control
- define on-policy vs. off-policy learning
- define a behaviour policy
- define exploring starts
- explain what problem exploring starts solve
- define an epsilon-soft policy
- explain what problem epsilon-soft policies solve

Logistics

- Late submissions for **Assignment #3** accepted until Friday at 11:59pm
- **Assignment #4** was released today!
    - Due **April 11** at 11:59pm

Previous Lecture Summary

- An **optimal policy** has higher state value than any other policy **at every state**
- A policy's state-value function can be computed by **iterating** an **expected update** based on the Bellman equation
- Given any policy π, we can compute a **greedy improvement** π' by choosing highest expected value action based on v_π
- **Policy iteration:** Repeat:
  Greedy improvement using v_π, then recompute v_π
- **Value iteration:** Repeat:
  Recompute v_π **by assuming** greedy improvement at every update

Recap: In-Place Iterative Policy Evaluation

Iterative Policy Evaluation, for estimating $V \approx v_\pi$

Input π, the policy to be evaluated
Algorithm parameter: a small threshold $\theta > 0$ determining accuracy of estimation
Initialize $V(s)$, for all $s \in \mathcal{S}^+$, arbitrarily except that $V(terminal) = 0$

Loop:
  $\Delta \leftarrow 0$
  Loop for each $s \in \mathcal{S}$:
    $v \leftarrow V(s)$
    $V(s) \leftarrow \sum_a \pi(a|s) \sum_{s',r} p(s', r \,|\, s, a)\big[r + \gamma V(s')\big]$
    $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$

- These are **expected updates:** Based on a weighted average (expectation) of **all possible next states**

Recap: Policy Improvement Theorem

Theorem:
Let π and π' be any pair of deterministic policies.

If $q_\pi(s, \pi'(s)) \geq v_\pi(s) \quad \forall s \in \mathcal{S}$,

then $v_{\pi'}(s) \geq v_\pi(s) \quad \forall s \in \mathcal{S}$.

If you are never worse off **at any state** by following π' for **one step** and then following π forever after, then following π' **forever** has a higher expected value **at every state**.

Policy Iteration

$$\pi_0 \xrightarrow{\text{E}} v_{\pi_0} \xrightarrow{\text{I}} \pi_1 \xrightarrow{\text{E}} v_{\pi_1} \xrightarrow{\text{I}} \pi_2 \xrightarrow{\text{E}} \cdots \xrightarrow{\text{I}} \pi_* \xrightarrow{\text{E}} v_*$$

Policy Iteration (using iterative policy evaluation) for estimating $\pi \approx \pi_*$

1. Initialization
   $V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

2. Policy Evaluation
   Loop:
   $\Delta \leftarrow 0$
   Loop for each $s \in \mathcal{S}$:
   $v \leftarrow V(s)$
   $V(s) \leftarrow \sum_{s',r} p(s', r \mid s, \pi(s))\big[r + \gamma V(s')\big]$
   $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
   until $\Delta < \theta$ (a small positive number determining the accuracy of estimation)

3. Policy Improvement
   policy-stable $\leftarrow$ *true*
   For each $s \in \mathcal{S}$:
   old-action $\leftarrow \pi(s)$
   $\pi(s) \leftarrow \arg\max_a \sum_{s',r} p(s', r \mid s, a)\big[r + \gamma V(s')\big]$
   If *old-action* $\neq \pi(s)$, then *policy-stable* $\leftarrow$ *false*
   If *policy-stable*, then stop and return $V \approx v_*$ and $\pi \approx \pi_*$; else go to 2

This is a lot of iterations!
Is it necessary to run to completion?

Value Iteration

Value iteration interleaves the estimation and improvement steps:

$$
\begin{aligned}
v_{k+1}(s) &\doteq \max_a \mathbb{E}\left[R_{t+1} + \gamma v_k(S_{t+1}) \,|\, S_t = s, A_t = a\right] \\
&= \max_a \sum_{s',r} p(s',r|s,a)\left[r + \gamma v_k(s')\right]
\end{aligned}
$$

Value Iteration, for estimating $\pi \approx \pi_*$

Algorithm parameter: a small threshold $\theta > 0$ determining accuracy of estimation
Initialize $V(s)$, for all $s \in \mathcal{S}^+$, arbitrarily except that $V(terminal) = 0$

Loop:
| $\Delta \leftarrow 0$
| Loop for each $s \in \mathcal{S}$:
| $v \leftarrow V(s)$
| $V(s) \leftarrow \max_a \sum_{s',r} p(s',r|s,a)\left[r + \gamma V(s')\right]$
| $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$

Output a deterministic policy, $\pi \approx \pi_*$, such that
  $\pi(s) = \arg\max_a \sum_{s',r} p(s',r|s,a)\left[r + \gamma V(s')\right]$

Example: Blackjack

- Player gets two cards, dealer gets 1
- Player can `hit` (get a new card) as many times as they like, or `stick` (stop hitting)
- After the player is done, the dealer hits / sticks according to a fixed rule
- Whoever has the most points (sum of card values) wins
- But, if you have more than 21 points, you **lose immediately** ("bust")

Simulating Blackjack

- Given a policy for the player, it is **very easy** to simulate a game of Blackjack
  - (Treat both the cards and the dealer as part of the environment)
- **Question:** Is it easy to **compute** the full **dynamics**?
- **Question:** Is it easy to run **iterative policy evaluation**?

Experience vs. Expectation

- In order to compute **expected updates**, we need to know the exact **probability** of **every** possible transition

- Often we don't have access to the full probability distribution, but we do have access to **samples of experience**

1. **Actual experience:** We want to learn based on interactions with a **real environment**, without knowing its dynamics

2. **Simulated experience:** We can **simulate** the dynamics, but we don't have an **explicit representation** of transition probabilities, or there are **too many states**

Monte Carlo Estimation

- Instead of estimating expectations by a **weighted sum** over **all possibilities**, estimate expectation by **averaging** over a **sample** drawn from the distribution:

$$\mathbb{E}[X] = \sum_x f(x)x \approx \frac{1}{n}\sum_{i=1}^{n} x_i \quad \text{where } x_i \sim f$$

Monte Carlo Prediction

- Use a **large sample** of **episodes** generated by a policy π to estimate the state-values $v_\pi(s)$ for each state s
  - We will consider only **episodic** tasks for now
- **Question:** What is the **return** G_t for state $S_t = s$ in a given episode?
- We can estimate the expected return $v_\pi(s) = \mathbb{E}[G_t \mid S_t = s]$ by averaging the returns for that state in every episode containing a visit to s

First-visit Monte Carlo Prediction

First-visit MC prediction, for estimating $V \approx v_\pi$

Input: a policy π to be evaluated

Initialize:

$V(s) \in \mathbb{R}$, arbitrarily, for all $s \in \mathcal{S}$

$Returns(s) \leftarrow$ an empty list, for all $s \in \mathcal{S}$

Loop forever (for each episode):

Generate an episode following π: $S_0, A_0, R_1, S_1, A_1, R_2, \ldots, S_{T-1}, A_{T-1}, R_T$

$G \leftarrow 0$

Loop for each step of episode, $t = T-1, T-2, \ldots, 0$:

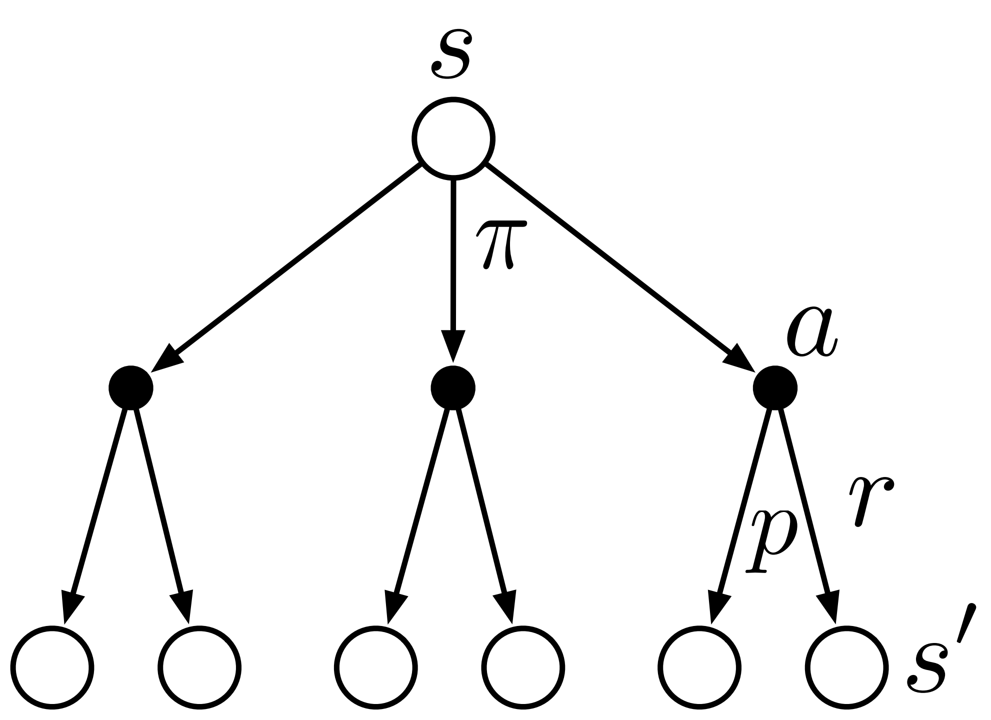
$G \leftarrow \gamma G + R_{t+1}$

Unless S_t appears in $S_0, S_1, \ldots, S_{t-1}$:

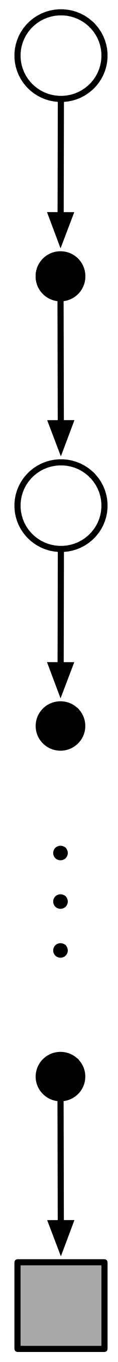
Append G to $Returns(S_t)$

$V(S_t) \leftarrow \text{average}(Returns(S_t))$

Monte Carlo vs. Dynamic Programming



- **Iterative policy evaluation** uses the estimates of the **next state's** value to update the value of this state
  - Only needs to compute a **single transition** to update a state's estimate
- **Monte Carlo** estimate of each state's value is **independent** from estimates of **other states'** values
  - Needs the **entire episode** to compute an update
  - Can focus on evaluating a **subset of states** if desired



Control vs. Prediction

- **Prediction:** estimate the value of states and/or actions given some **fixed policy** π
- **Control:** estimate an **optimal policy**

Estimating Action Values

- When we know the **dynamics** $p(s', r \mid s, a)$, an estimate of **state values** is sufficient to determine a good **policy**:
  - Choose the action that gives the best combination of **reward** and **next-state value**:

$$\hat{a}^* = \arg\max_{a \in \mathcal{A}} \sum_{s',r} p(s', r \mid s, a)[r + \gamma \hat{v}(s')]$$

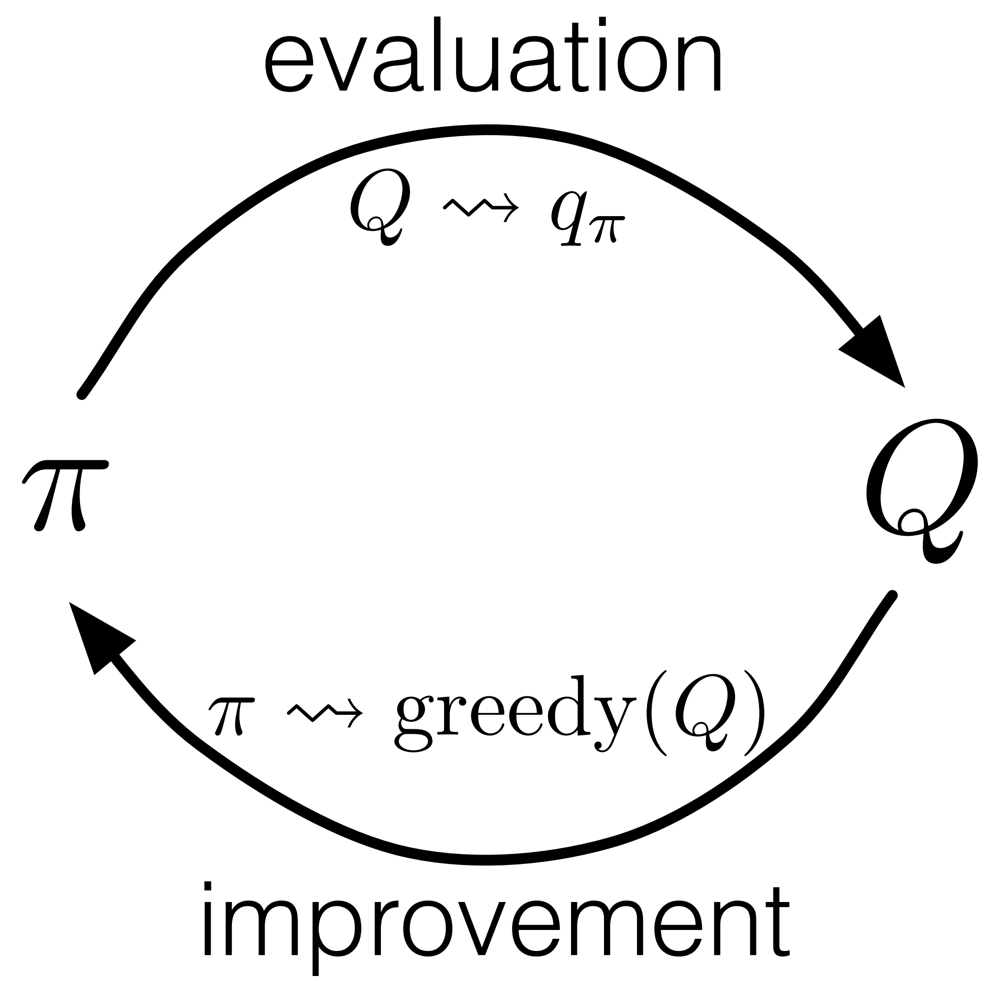
  - This is why Value Iteration only explicitly estimates state values, not policy
- If we *don't* know the dynamics, state values are **not enough**
  - To estimate a good policy, we need an **explicit** estimate of **action values**

Exploring Starts for Action-Value Estimation

- We can just run first-visit Monte Carlo and approximate the returns to each **state-action pair**
- **Question:** What do we do about state-action pairs that are **never visited**?
  - If the current policy π never selects an action a from a state s, then Monte Carlo can't estimate its value
- **Exploring starts assumption:**
  - Every episode **starts** at a random state-action pair S_0, A_0
  - **Every pair** has a positive probability of being selected for a start

Monte Carlo Control

Monte Carlo control can be used for **policy iteration**:



$$\pi_0 \xrightarrow{\text{E}} q_{\pi_0} \xrightarrow{\text{I}} \pi_1 \xrightarrow{\text{E}} q_{\pi_1} \xrightarrow{\text{I}} \pi_2 \xrightarrow{\text{E}} \cdots \xrightarrow{\text{I}} \pi_* \xrightarrow{\text{E}} q_*$$

Monte Carlo Control with Exploring Starts

Monte Carlo ES (Exploring Starts), for estimating $\pi \approx \pi_*$

Initialize:

$\pi(s) \in \mathcal{A}(s)$ (arbitrarily), for all $s \in \mathcal{S}$

$Q(s,a) \in \mathbb{R}$ (arbitrarily), for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$

$Returns(s,a) \leftarrow$ empty list, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$

Loop forever (for each episode):

Choose $S_0 \in \mathcal{S}$, $A_0 \in \mathcal{A}(S_0)$ randomly such that all pairs have probability > 0

Generate an episode from S_0, A_0, following π: $S_0, A_0, R_1, \ldots, S_{T-1}, A_{T-1}, R_T$

$G \leftarrow 0$

Loop for each step of episode, $t = T-1, T-2, \ldots, 0$:

$G \leftarrow \gamma G + R_{t+1}$

Unless the pair S_t, A_t appears in $S_0, A_0, S_1, A_1 \ldots, S_{t-1}, A_{t-1}$:

Append G to $Returns(S_t, A_t)$

$Q(S_t, A_t) \leftarrow \text{average}(Returns(S_t, A_t))$

$\pi(S_t) \leftarrow \arg\max_a Q(S_t, a)$

Question: What unlikely assumptions does this rely upon?

ε-Soft Policies

- The **exploring starts** assumption requires that we see **every** state-action pair with positive probability
  - Even if π **never** chooses a from state s
- Another approach: Simply **force** π to (sometimes) choose a!
- An **ϵ-soft policy** is one for which $\pi(a \mid s) \geq \frac{\epsilon}{|\mathcal{A}(s)|} \quad \forall s, a$
- **Example: ϵ-greedy policy**

$$\pi(a \mid s) = \begin{cases} \frac{\epsilon}{|\mathcal{A}(s)|} & \text{if } a \notin \arg\max_a Q(s,a), \\ (1-\epsilon) + \frac{\epsilon}{|\mathcal{A}(s)|} & \text{otherwise.} \end{cases}$$

Monte Carlo Control w/out Exploring Starts

On-policy first-visit MC control (for ε-soft policies), estimates $\pi \approx \pi_*$

Algorithm parameter: small $\varepsilon > 0$

Initialize:

$\pi \leftarrow$ an arbitrary ε-soft policy

$Q(s, a) \in \mathbb{R}$ (arbitrarily), for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$

$Returns(s, a) \leftarrow$ empty list, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$

Repeat forever (for each episode):

Generate an episode following π: $S_0, A_0, R_1, \ldots, S_{T-1}, A_{T-1}, R_T$

$G \leftarrow 0$

Loop for each step of episode, $t = T-1, T-2, \ldots, 0$:

$G \leftarrow \gamma G + R_{t+1}$

Unless the pair S_t, A_t appears in $S_0, A_0, S_1, A_1 \ldots, S_{t-1}, A_{t-1}$:

Append G to $Returns(S_t, A_t)$

$Q(S_t, A_t) \leftarrow \text{average}(Returns(S_t, A_t))$

$A^* \leftarrow \arg\max_a Q(S_t, a)$ (with ties broken arbitrarily)

For all $a \in \mathcal{A}(S_t)$:

$$\pi(a|S_t) \leftarrow \begin{cases} 1 - \varepsilon + \varepsilon/|\mathcal{A}(S_t)| & \text{if } a = A^* \\ \varepsilon/|\mathcal{A}(S_t)| & \text{if } a \neq A^* \end{cases}$$

Monte Carlo Control w/out Exploring Starts

On-policy first-visit MC control (for ε-soft policies), estimates $\pi \approx \pi_*$

Algorithm parameter: small $\varepsilon > 0$

Initialize:

$\pi \leftarrow$ an arbitrary ε-soft policy

$Q(s,a) \in \mathbb{R}$ (arbitrarily), for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$

$Returns(s,a) \leftarrow$ empty list, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$

Repeat forever (for each episode):

Generate an episode following π: $S_0, A_0, R_1, \ldots, S_{T-1}, A_{T-1}, R_T$

$G \leftarrow 0$

Loop for each step of episode, $t = T-1, T-2, \ldots, 0$:

$G \leftarrow \gamma G + R_{t+1}$

Unless the pair S_t, A_t appears in $S_0, A_0, S_1, A_1 \ldots, S_{t-1}, A_{t-1}$:

Append G to $Returns(S_t, A_t)$

$Q(S_t, A_t) \leftarrow \text{average}(Returns(S_t, A_t))$

$A^* \leftarrow \arg\max_a Q(S_t, a)$ (with ties broken arbitrarily)

For all $a \in \mathcal{A}(S_t)$:

$$\pi(a|S_t) \leftarrow \begin{cases} 1 - \varepsilon + \varepsilon/|\mathcal{A}(S_t)| & \text{if } a = A^* \\ \varepsilon/|\mathcal{A}(S_t)| & \text{if } a \neq A^* \end{cases}$$

Question:

Will this procedure converge to the optimal policy π^*?

Why or why not?

Importance Sampling

- **Monte Carlo sampling:** use samples from the **target** distribution to estimate expectations
- **Importance sampling:** Use samples from **proposal** distribution to estimate expectations of **target** distribution by **reweighting** samples

$$\mathbb{E}[X] = \sum_x f(x)x = \sum_x \frac{g(x)}{g(x)} f(x)x = \sum_x g(x)\frac{f(x)}{g(x)}x \approx \frac{1}{n}\sum_{x_i \sim g} \boxed{\frac{f(x_i)}{g(x_i)}} x_i$$

Importance sampling ratio

Off-Policy Prediction via Importance Sampling

Definition:

Off-policy learning means using data generated by a **behaviour policy** to learn about a distinct **target policy**.

Proposal distribution (→ behaviour policy)

Target distribution (→ target policy)

Off-Policy Monte Carlo Prediction

- Generate episodes using **behaviour policy** b
- Take **weighted** average of returns to state s over all the episodes containing a visit to s to estimate $v_\pi(s)$
  - Weighed by **importance sampling ratio** of trajectory starting from $S_t = s$ until the end of the episode:

$$\rho_{t:T-1} \doteq \frac{\Pr[A_t, S_{t+1}, \ldots, S_T \mid S_t, A_{t:T-1} \sim \pi]}{\Pr[A_t, S_{t+1}, \ldots, S_T \mid S_t, A_{t:T-1} \sim b]}$$

Importance Sampling Ratios for Trajectories

- **Probability of a trajectory** $A_t, S_{t+1}, A_{t+1}, \ldots, S_T$ from S_t:

$$\Pr[A_t, S_{t+1}, \ldots, S_T \mid S_t, A_{t:T-1} \sim \pi] =$$

$$\pi(A_t \mid S_t) p(S_{t+1} \mid S_t, A_t) \pi(A_{t+1} \mid S_{t+1}) \ldots p(S_T \mid S_{T-1}, A_{T-1})$$

- **Importance sampling ratio** for a trajectory $A_t, S_{t+1}, A_{t+1}, \ldots, S_T$ from S_t:

$$\rho_{t:T-1} \doteq \frac{\prod_{k=t}^{T-1} \pi(A_k \mid S_k) \boxed{p(S_{k+1} \mid S_k, A_k)}}{\prod_{k=t}^{T-1} b(A_k \mid S_k) \boxed{p(S_{k+1} \mid S_k, A_k)}} = \frac{\prod_{k=t}^{T-1} \pi(A_k \mid S_k)}{\prod_{k=t}^{T-1} b(A_k \mid S_k)}$$

$= 1$

Ordinary vs. Weighted Importance Sampling

- **Ordinary importance sampling:**

$$V(s) \doteq \frac{1}{n}\sum_{i=1}^{n} \rho_{t(s,i):T(i)-1} G_{i,t}$$

- **Weighted importance sampling:**

$$V(s) \doteq \frac{\sum_{i=1}^{n} \rho_{t(s,i):T(i)-1} G_{i,t}}{\sum_{i=1}^{n} \rho_{t(s,i):T(i)-1}}$$

Example: Ordinary vs. Weighted Importance Sampling for Blackjack

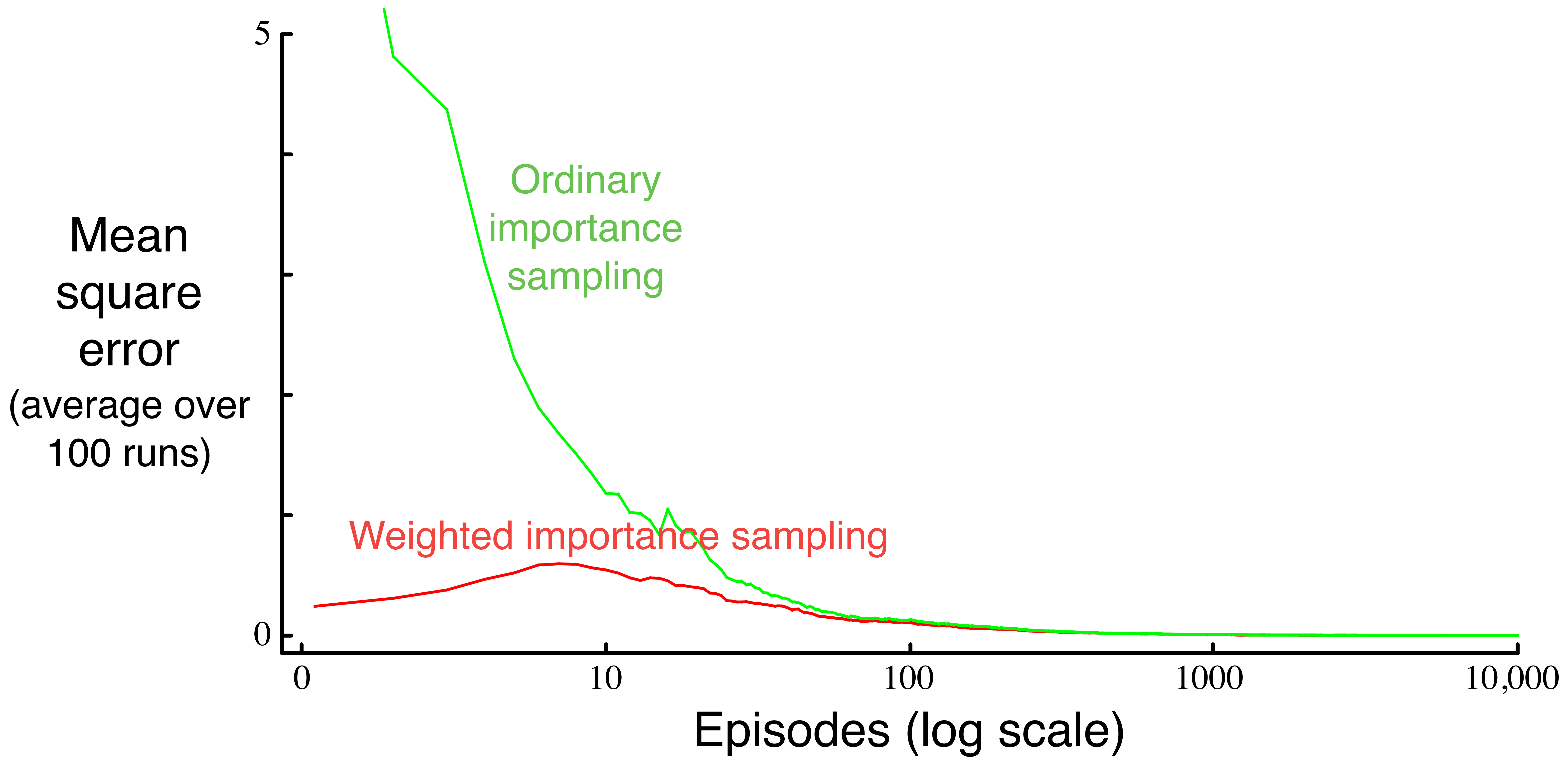


Figure 5.3: Weighted importance sampling produces lower error estimates of the value of a single blackjack state from off-policy episodes. ■

(Image: Sutton & Barto, 2018)

Off-Policy Monte Carlo **Prediction**

Off-policy MC prediction (policy evaluation) for estimating $Q \approx q_\pi$

Input: an arbitrary target policy π

Initialize, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$:

    $Q(s, a) \in \mathbb{R}$ (arbitrarily)

    $C(s, a) \leftarrow 0$

Loop forever (for each episode):

    $b \leftarrow$ any policy with coverage of π

    Generate an episode following b: $S_0, A_0, R_1, \ldots, S_{T-1}, A_{T-1}, R_T$

    $G \leftarrow 0$

    $W \leftarrow 1$

    Loop for each step of episode, $t = T-1, T-2, \ldots, 0$, while $W \neq 0$:

        $G \leftarrow \gamma G + R_{t+1}$

        $C(S_t, A_t) \leftarrow C(S_t, A_t) + W$

        $Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \frac{W}{C(S_t, A_t)} [G - Q(S_t, A_t)]$

        $W \leftarrow W \frac{\pi(A_t|S_t)}{b(A_t|S_t)}$

Off-Policy Monte Carlo Control

Off-policy MC control, for estimating $\pi \approx \pi_*$

Initialize, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$:
    $Q(s,a) \in \mathbb{R}$ (arbitrarily)
    $C(s,a) \leftarrow 0$
    $\pi(s) \leftarrow \arg\max_a Q(s,a)$ (with ties broken consistently)

Loop forever (for each episode):
    $b \leftarrow$ any soft policy
    Generate an episode using b: $S_0, A_0, R_1, \ldots, S_{T-1}, A_{T-1}, R_T$
    $G \leftarrow 0$
    $W \leftarrow 1$
    Loop for each step of episode, $t = T-1, T-2, \ldots, 0$:
        $G \leftarrow \gamma G + R_{t+1}$
        $C(S_t, A_t) \leftarrow C(S_t, A_t) + W$
        $Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \frac{W}{C(S_t, A_t)} [G - Q(S_t, A_t)]$
        $\pi(S_t) \leftarrow \arg\max_a Q(S_t, a)$ (with ties broken consistently)
        If $A_t \neq \pi(S_t)$ then exit inner Loop (proceed to next episode)
        $W \leftarrow W \frac{1}{b(A_t|S_t)}$

Off-Policy Monte Carlo Control

Off-policy MC control, for estimating $\pi \approx \pi_*$

Initialize, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$:
  $Q(s,a) \in \mathbb{R}$ (arbitrarily)
  $C(s,a) \leftarrow 0$
  $\pi(s) \leftarrow \arg\max_a Q(s,a)$ (wi

Loop forever (for each episode):
  $b \leftarrow$ any soft policy
  Generate an episode using b:
  $G \leftarrow 0$
  $W \leftarrow 1$
  Loop for each step of episode, $t = T-1, T-2, \ldots, 0$:
    $G \leftarrow \gamma G + R_{t+1}$
    $C(S_t, A_t) \leftarrow C(S_t, A_t) + W$
    $Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \frac{W}{C(S_t,A_t)} [G - Q(S_t, A_t)]$
    $\pi(S_t) \leftarrow \arg\max_a Q(S_t, a)$ (with ties broken consistently)
    If $A_t \neq \pi(S_t)$ then exit inner Loop (proceed to next episode)
    $W \leftarrow W \frac{1}{b(A_t|S_t)}$

$$Q_n = \frac{\sum_{i=1}^{n} W_i G_i}{\sum_{i=1}^{n} W_i} = \frac{\sum_{i=1}^{n} W_i G_i}{C - W}$$

$$Q_{n+1} = \frac{\sum_{i=1}^{n+1} W_i G_i}{\sum_{i=1}^{n+1} W_i} = \frac{(C - W)Q_n + WG}{C}$$

$$= \frac{C}{C}Q_n - \frac{W}{C}Q_n + \frac{W}{C}G = Q_n + \frac{W}{C}\left[G - Q_n\right]$$

Questions:

1. Will this procedure converge to the optimal policy π^*?
2. Why do we break when $A_t \neq \pi(S_t)$?
3. Why do the weights W not involve $\pi(A_t \mid S_t)$?

Summary

- **Monte Carlo estimation** estimates values by averaging returns over **sample episodes**
  - Does not require access to full model of **dynamics**
  - Does require access to an entire **episode** for each sample
- Estimating **action values** requires either **exploring starts** or a **soft policy** (e.g., ϵ-greedy)
- **Off-policy learning** is the estimation of value functions for a **target policy** based on episodes generated by a different **behaviour policy**
  - **Importance sampling** is one way to perform off-policy learning
  - **Weighted** importance sampling has lower **variance** than **ordinary** importance sampling
- **Off-policy control** is learning the **optimal policy** (target policy) using episodes from a **behaviour policy**