Optimality and Dynamic Programming

CMPUT 261: Introduction to Artificial Intelligence

S&B §3.6, §4.0-4.4

Lecture Outline

1. Recap & Logistics
2. Policy Evaluation
3. Optimality
4. Policy Improvement

After this lecture, you should be able to:

- justify why one policy is weakly better than another
- trace an execution of iterative policy evaluation
- state the Policy Improvement Theorem and describe why it is important
- trace an execution of the Value Iteration algorithm

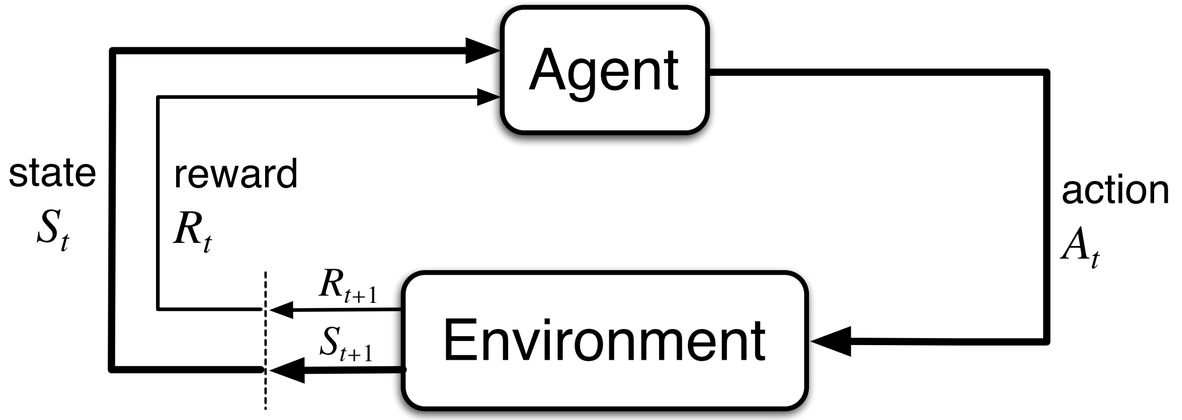
Logistics

- **Assignment #3** is **due tonight** at 11:59pm
  - No remarking or EAs for empty submissions
- **Assignment #4** will be released today
  - Due **Thursday December 4** at 11:59pm
- Reminder: TAs are available during labs to help

Recap: Markov Decision Process

At each time $t = 1,2,3,\ldots$

1. Agent receives input denoting **current state** S_t
2. Agent chooses **action** A_t
3. Next time step, agent receives **reward** R_{t+1} and **new state** S_{t+1}, chosen according to a distribution $p(s', r \mid s, a)$



There are two boxes labelled "Agent" and "Environment". There is an arrow from Agent to Environment labelled "action A_t". There are two arrows from Environment pointing to a vertical dashed line labelled "R_{t+1}" and "S_{t+1}". From the dashed line there are two arrows pointing back to Agent labelled "state S_t" and "reward R_t".

This interaction between agent and environment produces a **trajectory**:

$$S_0, A_0, R_1, S_1, A_1, R_2, S_2, A_2, R_3, \ldots$$

Recap: Policies & Value Functions

Policy: A function $\pi : \mathcal{S} \to \Delta(\mathcal{A})$ that maps states to a distribution over actions

- For every state s, agent following policy π will play action a with probability $\pi(a \mid s)$

State-value function:

$$v_\pi(s) \doteq \mathbb{E}_\pi[G_t \mid S_t = s]$$
$$= \mathbb{E}_\pi \left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \,\middle|\, S_t = s \right]$$

Action-value function:

$$q_\pi(s, a) \doteq \mathbb{E}_\pi[G_t \mid S_t = s, A_t = a]$$
$$= \mathbb{E}_\pi \left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \,\middle|\, S_t = s, A_t = a \right]$$

Recap: Bellman Equations

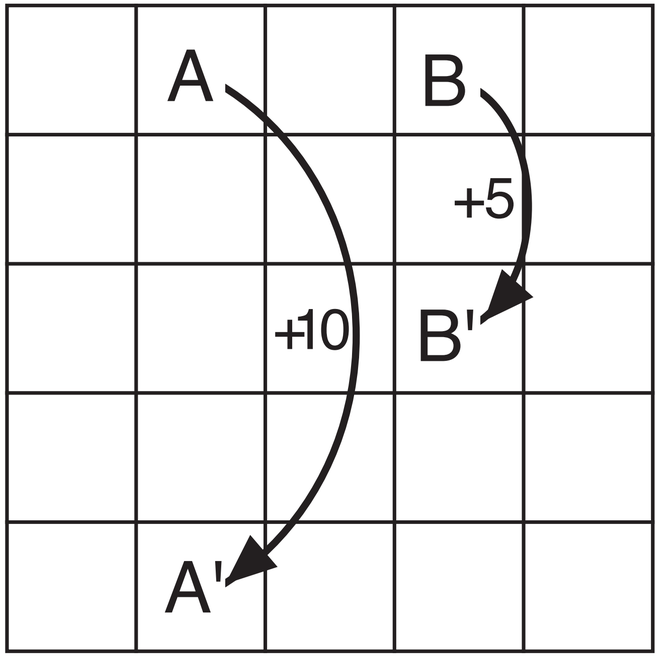
Value functions satisfy a **recursive consistency condition** called the **Bellman equation**:

$$
\begin{aligned}
v_\pi(s) &\doteq \mathbb{E}_\pi[G_t \,|\, S_t = s] \\
&= \mathbb{E}_\pi[R_{t+1} + \gamma G_{t+1} \,|\, S_t = s] \\
&= \sum_a \pi(a \,|\, s) \sum_{s'} \sum_r p(s', r \,|\, s, a) \Big[r + \gamma \mathbb{E}_\pi[G_{t+1} \,|\, S_{t+1} = s'] \Big] \\
&= \sum_a \pi(a \,|\, s) \sum_{s', r} p(s', r \,|\, s, a) \Big[r + \gamma v_\pi(s') \Big]
\end{aligned}
$$

- v_π is the **unique solution** to π's (state-value) Bellman equation
- There is also a Bellman equation for π's **action-value function**

Example: GridWorld

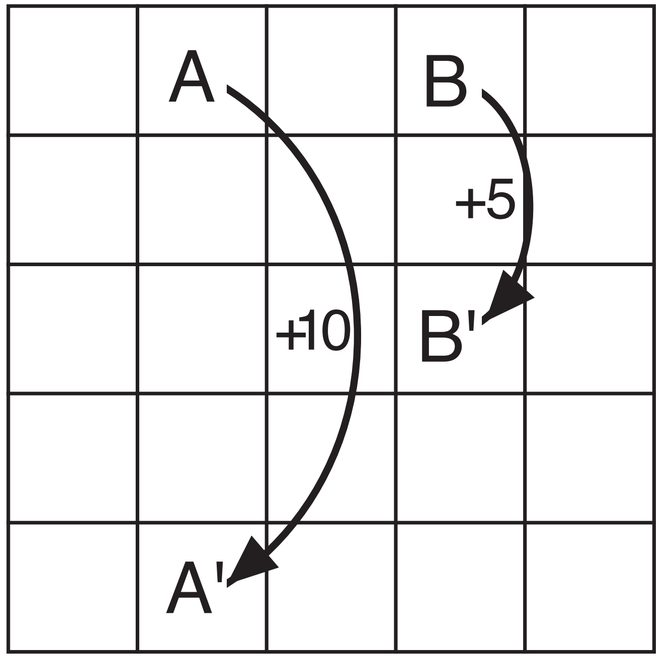
- At each cell, can go `north`, `south`, `east`, `west`
- Try to go off the **edge**: reward of **-1**
- Leaving state **A**: takes you to state **A′**, reward of **+10**
- Leaving state **B**: takes you to state **B′**, reward of **+5**



A grid with five rows and five columns. The cell in the top row, second column is labelled "A" and has an arrow labelled "+10" leading to the cell in the bottom row second column labelled "A'". The cell in the top row, fourth column has an arrow labelled "+5" leading to the cell in the third row, fourth column labelled "B'".

(Image: Sutton & Barto, 2018)

GridWorld



A grid with five rows and five columns. The cell in the top row, second column is labelled "A" and has an arrow labelled "+10" leading to the cell in the bottom row second column labelled "A'". The cell in the top row, fourth column has an arrow labelled "+5" leading to the cell in the third row, fourth column labelled "B'".

Reward dynamics

3.3	8.8	4.4	5.3	1.5
1.5	3.0	2.3	1.9	0.5
0.1	0.7	0.7	0.4	-0.4
-1.0	-0.4	-0.4	-0.6	-1.2
-1.9	-1.3	-1.2	-1.4	-2.0

A five-by-five grid with numbers in each cell; the top row has 3.3, 8.8, 4.4., 5.3, 1.5; the bottom row has -1.9, -1.3, -1.2, -1.4, -2.0.

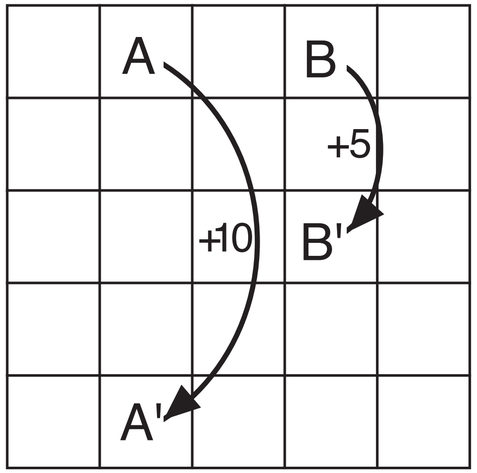
State-value function v_π for random policy

$$\pi(a \mid s) = 0.25$$

(Image: Sutton & Barto, 2018)

GridWorld with Bounds Checking

What about a policy where we never try to go over an edge?



A grid with five rows and five columns. The cell in the top row, second column is labelled "A" and has an arrow labelled "+10" leading to the cell in the bottom row second column labelled "A'". The cell in the top row, fourth column has an arrow labelled "+5" leading to the cell in the third row, fourth column labelled "B'".

Reward dynamics

3.3	8.8	4.4	5.3	1.5
1.5	3.0	2.3	1.9	0.5
0.1	0.7	0.7	0.4	-0.4
-1.0	-0.4	-0.4	-0.6	-1.2
-1.9	-1.3	-1.2	-1.4	-2.0

A five-by-five grid with numbers in each cell; the top row has 3.3, 8.8, 4.4., 5.3, 1.5; the bottom row has -1.9, -1.3, -1.2, -1.4, -2.0.

State-value function v_π for random policy $\pi(a \mid s) = 0.25$

6.7	10.8	6.4	6.7	4.3
4.2	4.7	3.7	3.4	2.8
2.4	2.4	2.1	1.9	1.7
1.5	1.4	1.3	1.2	1.1
1.1	1.0	0.9	0.9	0.9

State-value function v_{π^B} for **bounded** random policy π^B

Policy Evaluation

Question: How can we **compute** v_π?

1. We know that v_π is the unique solution to the Bellman equations, so we could just **solve them** (treating $v_\pi(s_1), \ldots, v_\pi(s_{|\mathcal{S}|})$ as variables)
    - but that is tedious and annoying and slow
      (it's a system of $|\mathcal{S}|$ linear equations in $|\mathcal{S}|$ unknowns)
    - Also requires a complete model of the dynamics
2. **Iterative policy evaluation**
    - Takes advantage of the recursive formulation

Iterative Policy Evaluation

- Iterative policy evaluation uses the Bellman equation as an **update rule**:

$$
\begin{aligned}
v_{k+1}(s) &\doteq \mathbb{E}_\pi[R_{t+1} + \gamma v_k(S_{t+1}) \,|\, S_t = s] \\
&= \sum_a \pi(a \,|\, s) \sum_{s', r} p(s', r \,|\, s, a) \Big[r + \gamma v_k(s') \Big]
\end{aligned}
$$

- v_π is a **fixed point** of this update, by definition
- Furthermore, starting from an **arbitrary** v_0, the sequence $\{v_k\}$ will **converge** to v_π as $k \to \infty$
  - *(nontrivial to prove)*

In-Place Iterative Policy Evaluation

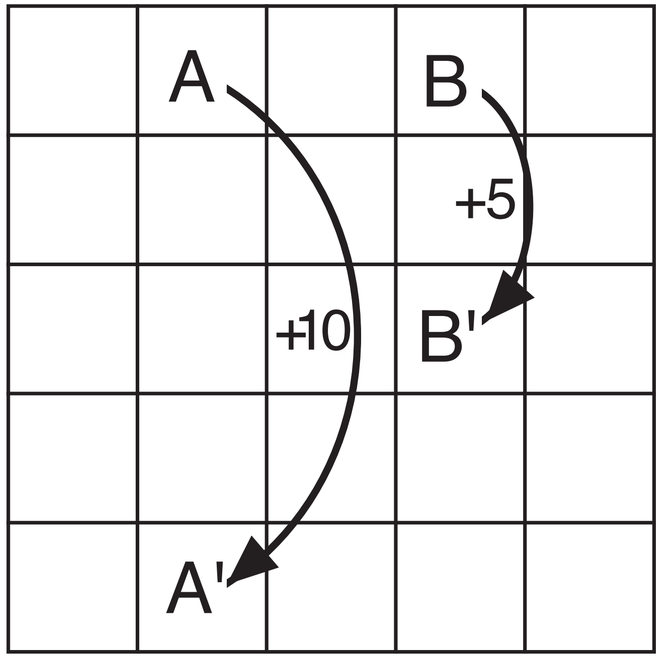
Iterative Policy Evaluation, for estimating $V \approx v_\pi$

Input π, the policy to be evaluated
Algorithm parameter: a small threshold $\theta > 0$ determining accuracy of estimation
Initialize $V(s)$, for all $s \in \mathcal{S}^+$, arbitrarily except that $V(terminal) = 0$

Loop:
  $\Delta \leftarrow 0$
  Loop for each $s \in \mathcal{S}$:
    $v \leftarrow V(s)$
    $V(s) \leftarrow \sum_a \pi(a|s) \sum_{s',r} p(s', r \,|\, s, a)\big[r + \gamma V(s')\big]$
    $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$

- The updates are **in-place**: we use new values for $V(s)$ **immediately** instead of waiting for the current sweep to complete (**why?**)
- These are **expected updates**: Based on a weighted average (expectation) of **all possible next states** (**instead of what?**)

Iterative Policy Evaluation



A grid with five rows and five columns. The cell in the top row, second column is labelled "A" and has an arrow labelled "+10" leading to the cell in the bottom row second column labelled "A'". The cell in the top row, fourth column has an arrow labelled "+5" leading to the cell in the third row, fourth column labelled "B'".

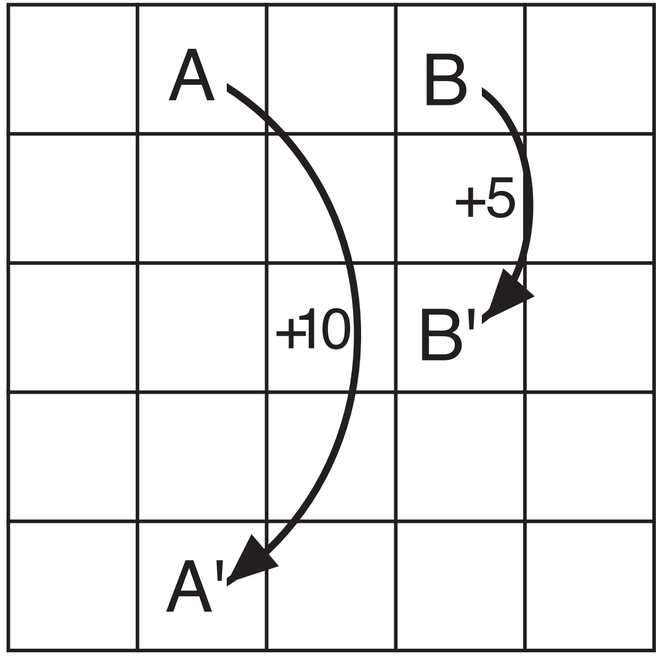
Reward dynamics

0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0

V at $k = 0$

Iterative Policy Evaluation

$$V(s_{1,1}) = \pi(\text{n})[-1 + \gamma V(s_{1,1})] + \pi(\text{w})[-1 + \gamma V(s_{1,1})] +$$
$$\pi(\text{s})[0 + \gamma V(s_{1,2})] + \pi(\text{e})[0 + \gamma V(s_{2,1})]$$
$$= 0.25(-1) + 0.25(-1) + 0.25(0) + 0.25(0)$$



A grid with five rows and five columns. The cell in the top row, second column is labelled "A" and has an arrow labelled "+10" leading to the cell in the bottom row second column labelled "A'". The cell in the top row, fourth column has an arrow labelled "+5" leading to the cell in the third row, fourth column labelled "B'".

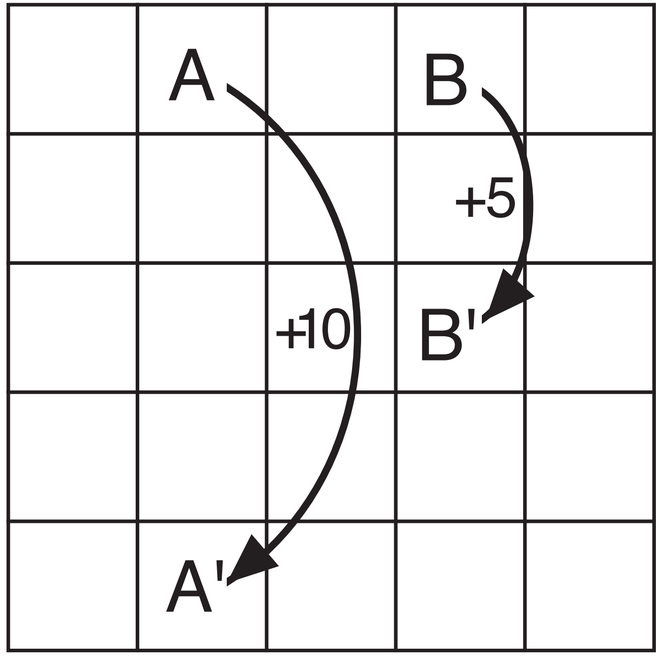
Reward dynamics

-0.5	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0

V at $k = 0$

Iterative Policy Evaluation

$$
\begin{aligned}
V(s_{1,2}) &= \pi(\text{n})[10 + \gamma \mathbf{V(s_{2,5})}] + \pi(\text{w})[10 + \gamma \mathbf{V(s_{2,5})}] + \\
&\quad \pi(\text{s})[10 + \gamma \mathbf{V(s_{2,5})}] + \pi(\text{e})[10 + \gamma \mathbf{V(s_{2,5})}] \\
&= 0.25[10 + 0.9(\mathbf{0})] + 0.25[10 + 0.9(\mathbf{0})] + \\
&\quad 0.25[10 + 0.9(\mathbf{0})] + 0.25[10 + 0.9(\mathbf{0})]
\end{aligned}
$$



A grid with five rows and five columns. The cell in the top row, second column is labelled "A" and has an arrow labelled "+10" leading to the cell in the bottom row second column labelled "A'". The cell in the top row, fourth column has an arrow labelled "+5" leading to the cell in the third row, fourth column labelled "B'".

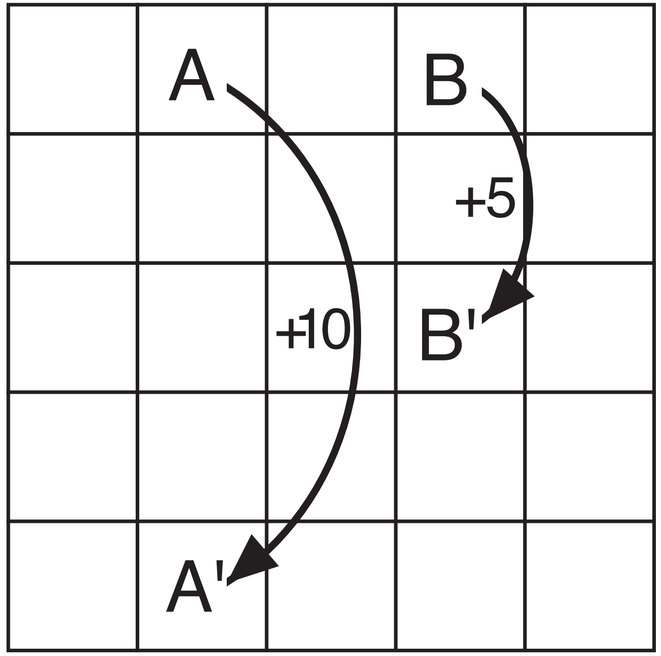
Reward dynamics

-0.5	10	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0

V at $k = 0$

Iterative Policy Evaluation

$$
\begin{aligned}
V(s_{3,1}) &= \pi(\text{n})[-1 + \gamma V(s_{3,1})] + \pi(\text{w})[-1 + \gamma \mathbf{V(s_{2,1})}] + \\
&\quad \pi(\text{s})[0 + \gamma V(s_{3,2})] + \pi(\text{e})[0 + \gamma V(s_{4,1})] \\
&= 0.25[-1 + 0.9(0)] + 0.25[0 + 0.9(\mathbf{10})] + \\
&\quad 0.25[0 + 0.9(0)] + 0.25[0 + 0.9(0)]
\end{aligned}
$$



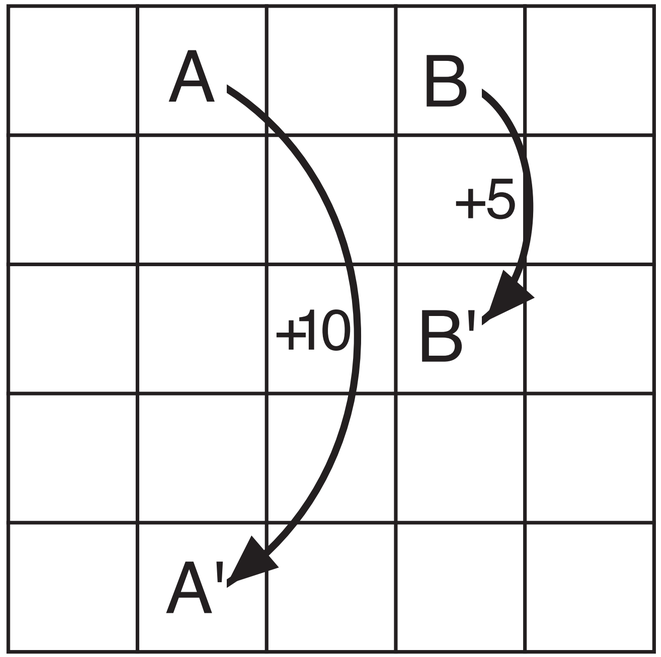
A grid with five rows and five columns. The cell in the top row, second column is labelled "A" and has an arrow labelled "+10" leading to the cell in the bottom row second column labelled "A'". The cell in the top row, fourth column has an arrow labelled "+5" leading to the cell in the third row, fourth column labelled "B'".

Reward dynamics

-0.5	10	2	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0

V at $k = 0$

Iterative Policy Evaluation in GridWorld



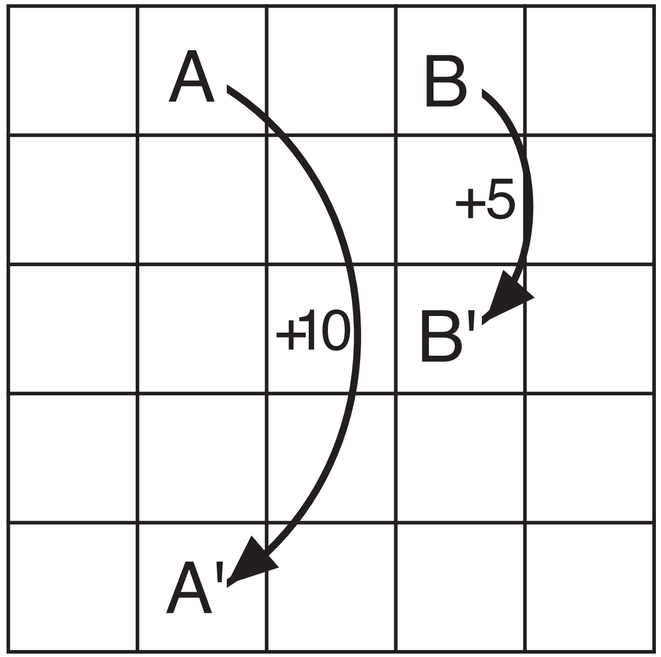
A grid with five rows and five columns. The cell in the top row, second column is labelled "A" and has an arrow labelled "+10" leading to the cell in the bottom row second column labelled "A'". The cell in the top row, fourth column has an arrow labelled "+5" leading to the cell in the third row, fourth column labelled "B'".

Reward dynamics

-0.5	10	2	5	0.6
-0.3	2.1	0.9	1.3	0.2
-0.3	0.4	0.3	0.4	-0.1
-0.3	0.0	0.0	0.1	-0.2
-0.5	-0.3	-0.3	-0.3	-0.6

V at $k = 1$

Iterative Policy Evaluation in GridWorld



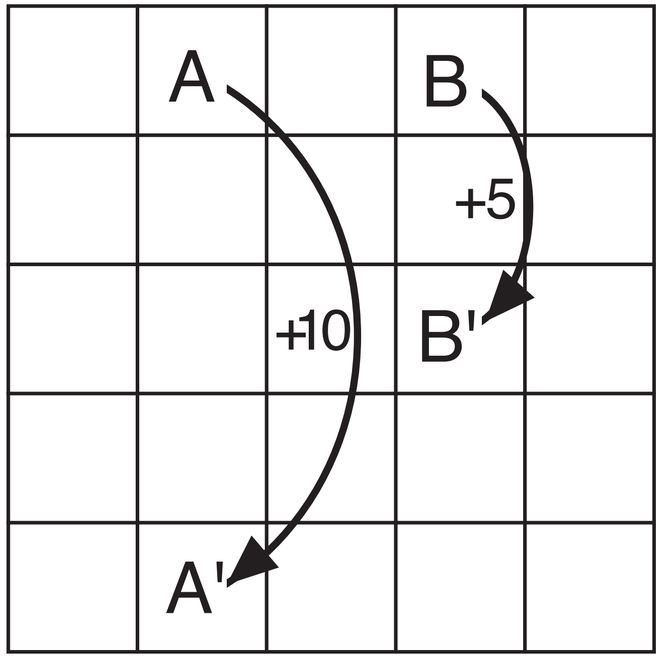
A grid with five rows and five columns. The cell in the top row, second column is labelled "A" and has an arrow labelled "+10" leading to the cell in the bottom row second column labelled "A'". The cell in the top row, fourth column has an arrow labelled "+5" leading to the cell in the third row, fourth column labelled "B'".

Reward dynamics

1.4	9.7	3.7	5.3	1.0
0.4	2.5	1.8	1.7	0.4
-0.2	0.6	0.6	0.5	-0.1
-0.5	0.0	0.0	0.0	-0.5
-1.0	-0.6	-0.5	-0.5	-1.0

V at $k = 2$

Iterative Policy Evaluation in GridWorld



A grid with five rows and five columns. The cell in the top row, second column is labelled "A" and has an arrow labelled "+10" leading to the cell in the bottom row second column labelled "A'". The cell in the top row, fourth column has an arrow labelled "+5" leading to the cell in the third row, fourth column labelled "B'".

Reward dynamics

3.4	8.9	4.5	5.3	1.5
1.6	3.0	2.3	1.9	0.6
0.1	0.8	0.7	0.4	-0.4
-1.0	-0.4	-0.3	-0.6	-1.2
-1.9	-1.3	-1.2	-1.4	-2.0

V at $k = 10\,000$

Optimality

- **Question:** What is an **optimal** policy?
- A policy π is (weakly) **better** than a policy π' if it is better **for all** $s \in \mathcal{S}$:

$$\pi \geq \pi' \iff v_\pi(s) \geq v_{\pi'}(s) \quad \forall s \in \mathcal{S}.$$

- An **optimal** policy π_* is weakly better than **every other policy**
  - **Question:** Is an optimal policy guaranteed to exist for a given MDP?
- All optimal policies share the **same state-value function**: (**why?**)

$$v_*(s) \doteq \max_\pi v_\pi(s)$$

- Also the same **action-value function**:

$$q_*(s, a) \doteq \max_\pi q_\pi(s, a)$$

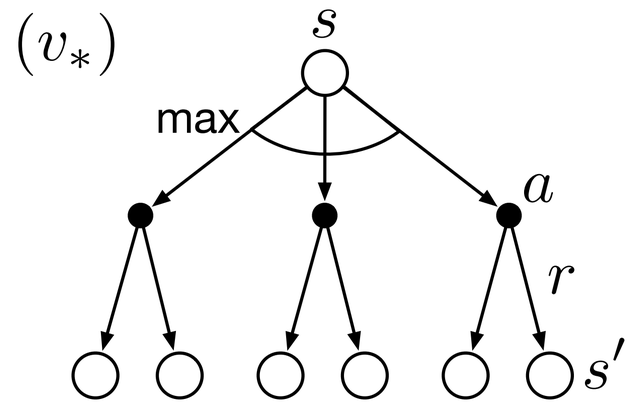
Bellman Optimality Equations

- v_* must satisfy the Bellman equation too
- In fact, it can be written in a special, **policy-free** way because we know that every state value is **maximized** by π_*:

$$
\begin{aligned}
v_*(s) &= \max_a q_{\pi_*}(s, a) \\
&= \max_a \mathbb{E}_{\pi_*}[G_t \mid S_t = s, A_t = a] \\
&= \max_a \mathbb{E}_{\pi_*}[R_{t+1} + \gamma G_{t+1} \mid S_t = s, A_t = a] \\
&= \max_a \mathbb{E}[R_{t+1} + \gamma v_*(S_{t+1}) \mid S_t = s, A_t = a] \\
&= \max_a \sum_{s', r} p(s', r \mid s, a)[r + \gamma v_*(s')]
\end{aligned}
$$

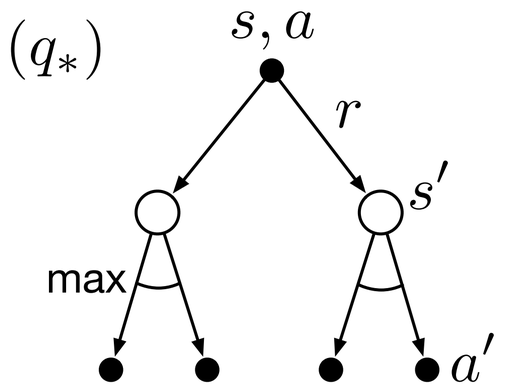
Bellman Optimality Equations

$$v_*(s) = \max_a \mathbb{E}[R_{t+1} + \gamma v_*(S_{t+1}) \,|\, S_t = s, A_t = a]$$
$$= \max_a \sum_{s',r} p(s', r \,|\, s, a)[r + \gamma v_*(s')]$$



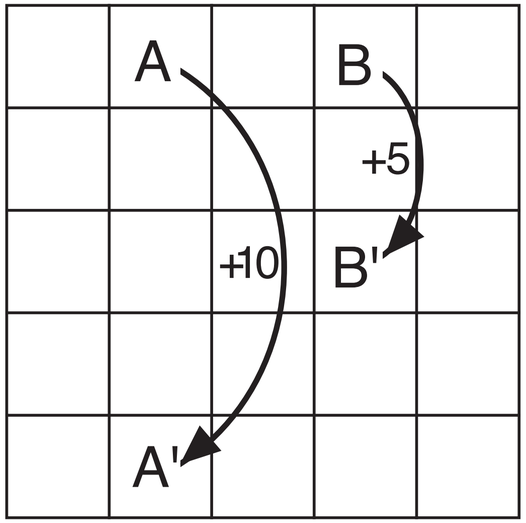
An open circle "s" has outgoing edges leading to three filled circles "a", each of which has two outgoing edges labelled "r" to open circles labelled "s'". The branches from "s" to the filled circles "a" are connected by an arc labelled "max".

$$q_*(s, a) = \mathbb{E}\left[R_{t+1} + \gamma \max_{a'} q_*(S_{t+1}, a') \,\middle|\, S_t = s, A_t = a\right]$$
$$= \sum_{s',r} p(s', r \,|\, s, a)\left[r + \gamma \max_{a'} q_*(s', a')\right]$$



A filled circle labelled "s,a" has outgoing edges labelled "r" to two open circles labelled "s'". Each "s'" circle has two edges leading to filled circles labelled "a'"; these two edges are joined by an arc labelled "max".

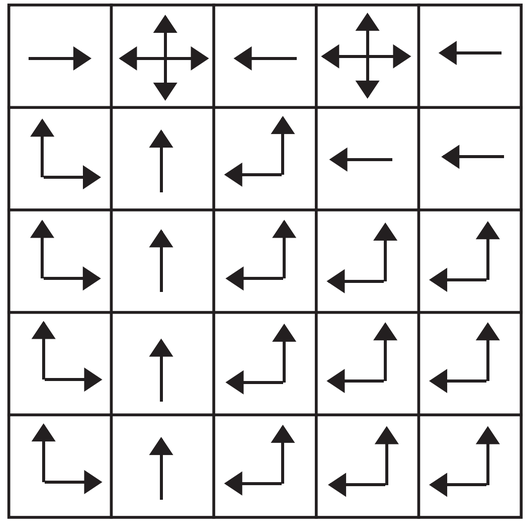
Optimal GridWorld



Gridworld

22.0	24.4	22.0	19.4	17.5
19.8	22.0	19.8	17.8	16.0
17.8	19.8	17.8	16.0	14.4
16.0	17.8	16.0	14.4	13.0
14.4	16.0	14.4	13.0	11.7

v_*



π_*

The reward dynamics for gridworld, a table of the optimal state values, and a grid showing the actions chosen by the optimal policy π_*.

Policy Improvement Theorem

Theorem:
Let π and π' be any pair of deterministic policies.

If $q_\pi(s, \pi'(s)) \geq v_\pi(s) \quad \forall s \in \mathcal{S}$,

then $v_{\pi'}(s) \geq v_\pi(s) \quad \forall s \in \mathcal{S}$.

If you are never worse off **at any state** by following π' for **one step** and then following π forever after, then following π' **forever** has a higher expected value **at every state**.

Policy Improvement Theorem Proof

$$
\begin{aligned}
v_\pi(s) &\le q_\pi(s, \pi'(s)) \qquad \forall s \\
&= \mathbb{E}_\pi[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s, A_t = \pi'(s)] \\
&= \mathbb{E}_{\pi'}[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s] \\
&\le \mathbb{E}_{\pi'}[R_{t+1} + \gamma q_\pi(S_{t+1}, \pi'(S_{t+1})) \mid S_t = s] \\
&= \mathbb{E}_{\pi'}[R_{t+1} + \gamma \mathbb{E}_{\pi'}[R_{t+2} + \gamma v_\pi(S_{t+2}) \mid S_{t+1}, A_{t+1} = \pi'(S_{t+1})] \mid S_t = s] \\
&= \mathbb{E}_{\pi'}[R_{t+1} + \gamma \mathbb{E}_{\pi'}[R_{t+2}] + \gamma^2 \mathbb{E}_{\pi'}[v_\pi(S_{t+2})] \mid S_t = s] \\
&= \mathbb{E}_{\pi'}[R_{t+1} + \gamma R_{t+2} + \gamma^2 v_\pi(S_{t+2}) \mid S_t = s] \\
&\le \mathbb{E}_{\pi'}[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 v_\pi(S_{t+3}) \mid S_t = s] \\
&\vdots \\
&\le \mathbb{E}_{\pi'}[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 R_{t+4} + \cdots \mid S_t = s] \\
&= v_{\pi'}(s) \,.
\end{aligned}
$$

Greedy Policy Improvement

Given any policy π, we can construct a new greedy policy π' that is guaranteed to be **at least as good**:

$$\begin{aligned}\pi'(s) &\doteq \arg\max_a q_\pi(s,a) \\ &= \arg\max_a \mathbb{E}[R_{t+1} + \gamma v_\pi(S_{T+1}) \,|\, S_t = s, A_t = a] \\ &= \arg\max_a \sum_{s',r} p(s', r \,|\, s, a)\left[r + \gamma v_\pi(s')\right] .\end{aligned}$$

- If this new policy is not **strictly** better than the old policy, then $v_\pi(s) = v_{\pi'}(s)$ for all $s \in \mathcal{S}$ (**why?**)
- Also means that the new (and old) policies are **optimal** (**why?**)

Policy Iteration

$$\pi_0 \xrightarrow{\text{E}} v_{\pi_0} \xrightarrow{\text{I}} \pi_1 \xrightarrow{\text{E}} v_{\pi_1} \xrightarrow{\text{I}} \pi_2 \xrightarrow{\text{E}} \cdots \xrightarrow{\text{I}} \pi_* \xrightarrow{\text{E}} v_*$$

Policy Iteration (using iterative policy evaluation) for estimating $\pi \approx \pi_*$

1. Initialization
   $V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

2. Policy Evaluation
   Loop:
   $\Delta \leftarrow 0$
   Loop for each $s \in \mathcal{S}$:
   $v \leftarrow V(s)$
   $V(s) \leftarrow \sum_{s',r} p(s', r \mid s, \pi(s)) \big[r + \gamma V(s')\big]$
   $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
   until $\Delta < \theta$ (a small positive number determining the accuracy of estimation)

3. Policy Improvement
   policy-stable $\leftarrow$ *true*
   For each $s \in \mathcal{S}$:
   old-action $\leftarrow \pi(s)$
   $\pi(s) \leftarrow \arg\max_a \sum_{s',r} p(s', r \mid s, a) \big[r + \gamma V(s')\big]$
   If *old-action* $\neq \pi(s)$, then *policy-stable* $\leftarrow$ *false*
   If *policy-stable*, then stop and return $V \approx v_*$ and $\pi \approx \pi_*$; else go to 2

This is a lot of iterations! Is it necessary to run to completion?

Value Iteration

Value iteration interleaves the estimation and improvement steps:

$$v_{k+1}(s) \doteq \max_a \mathbb{E}\left[R_{t+1} + \gamma v_k(S_{t+1}) \,|\, S_t = s, A_t = a\right]$$
$$= \max_a \sum_{s',r} p(s', r \,|\, s, a)\left[r + \gamma v_k(s')\right]$$

Value Iteration, for estimating $\pi \approx \pi_*$

Algorithm parameter: a small threshold $\theta > 0$ determining accuracy of estimation
Initialize $V(s)$, for all $s \in \mathcal{S}^+$, arbitrarily except that $V(terminal) = 0$

Loop:
| $\Delta \leftarrow 0$
| Loop for each $s \in \mathcal{S}$:
| $v \leftarrow V(s)$
| $V(s) \leftarrow \max_a \sum_{s',r} p(s', r \,|\, s, a)\left[r + \gamma V(s')\right]$
| $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$

Output a deterministic policy, $\pi \approx \pi_*$, such that
$\pi(s) = \arg\max_a \sum_{s',r} p(s', r \,|\, s, a)\left[r + \gamma V(s')\right]$

Summary

- An **optimal policy** has higher state value than any other policy **at every state**
- A policy's state-value function can be computed by **iterating** an **expected update** based on the Bellman equation
- Given any policy π, we can compute a **greedy improvement** π' by choosing highest expected value action based on v_π
- **Policy iteration:** Repeat:

  Greedy improvement using v_π, then recompute v_π
- **Value iteration:** Repeat:

  Recompute v_π **by assuming** greedy improvement at every update