Game Theory for Sequential Interactions

CMPUT 261: Introduction to Artificial Intelligence

S&LB §5.0-5.2.2

Lecture Outline

1. Recap & Logistics

2. Perfect Information Games

3. Backward Induction

After this lecture, you should be able to:

- trace an execution of backward induction

- identify the pure strategies in an extensive form game

- define the induced normal form

- construct the induced normal form when given an extensive form game

- identify a maxmin strategy in an extensive form game

Logistics

- **Assignment #4** is due December 7 (TONIGHT) at 11:59pm
  - Late submissions for 20% deduction until **December 11** at 11:59pm

- **SPOT** (formerly USRI) surveys are now available
  - Available until **December 10** at 11:59pm
  - You should have gotten an email
  - *Please do fill one out*, URL here: https://p20.courseval.net/etw/ets/et.asp?nxappid=UA2&nxmid=start

- **Final exam** is **Thursday, December 14**
  - in the Education Gym
  - At 9am *(half an hour earlier than lecture)*
  - Format: Like midterm, but longer
  - Material: *EVERYTHING* (but more focus on post-midterm material)

Recap: Game Theory

- Game theory studies the **interactions of rational agents**

    - Canonical representation is the **normal form game**

- Game theory uses **solution concepts** rather than optimal behaviour

    - "Optimal behaviour" is not clear-cut in multiagent settings

    - **Pareto optimal:** no agent can be made better off without making some other agent worse off

    - **Nash equilibrium:** no agent regrets their strategy given the choice of the other agents' strategies

	Ballet	Soccer
Ballet	2, 1	0, 0
Soccer	0, 0	1, 2

Extensive Form Games

- Normal form games don't have any notion of **sequence**: all actions happen **simultaneously**

- The **extensive form** is a game representation that explicitly includes temporal structure (i.e., a **game tree**)
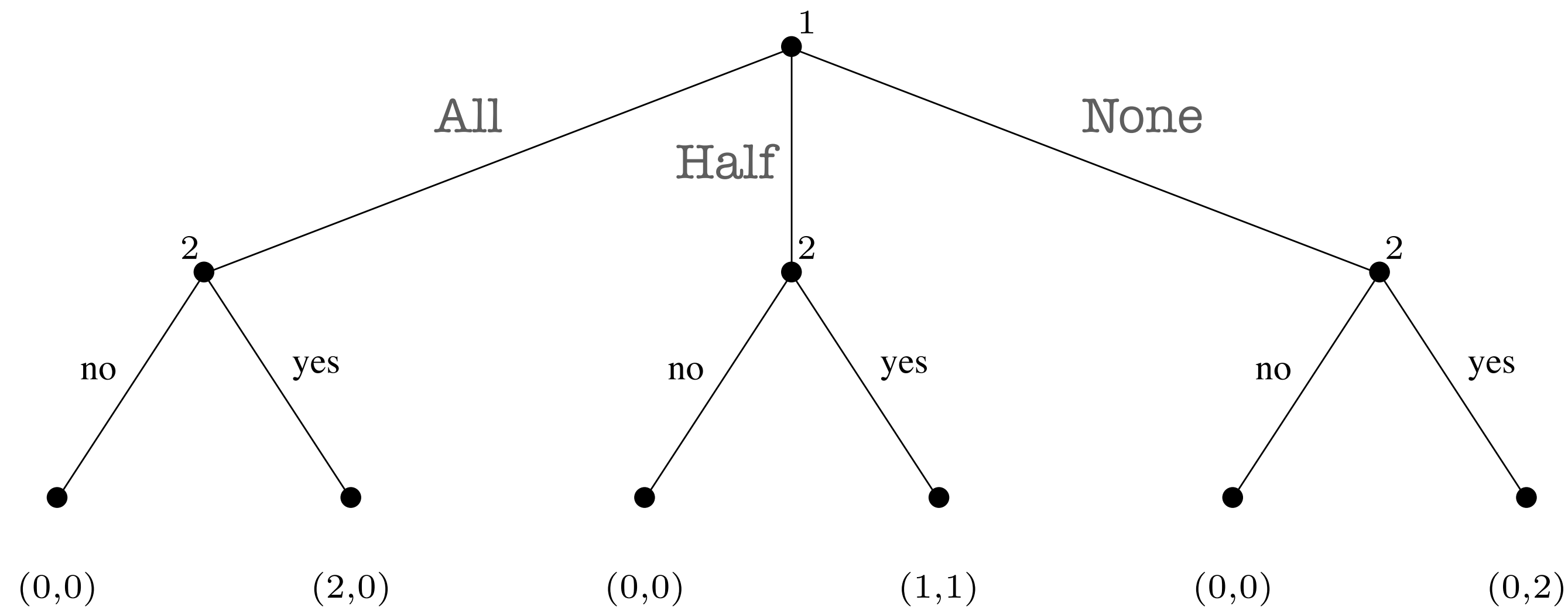
Perfect Information

There are two kinds of extensive form game:

1. **Perfect information:** Every agent sees all actions of the other players (including "Nature")

   - e.g.: Chess, checkers, Pandemic

2. **Imperfect information:** Some actions are hidden

   - Players may not know exactly where they are in the tree

   - e.g.: Poker, rummy, Scrabble
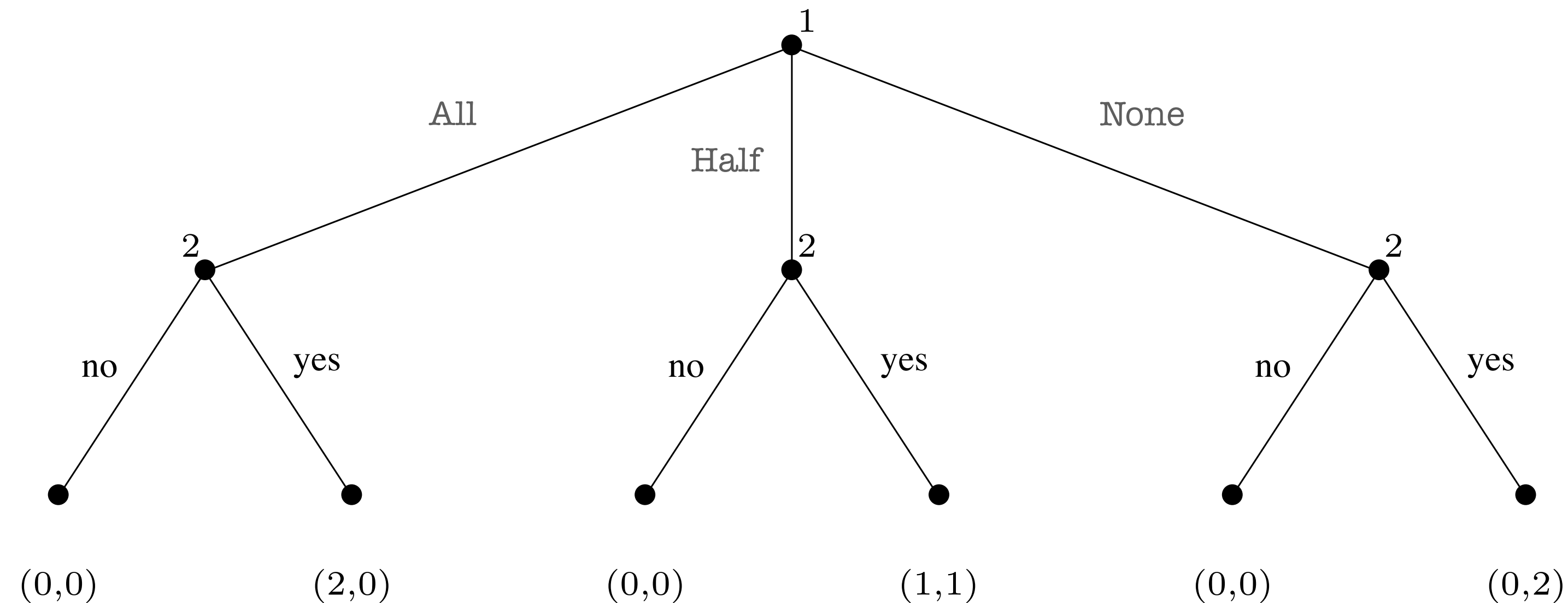
Perfect Information Extensive Form Game
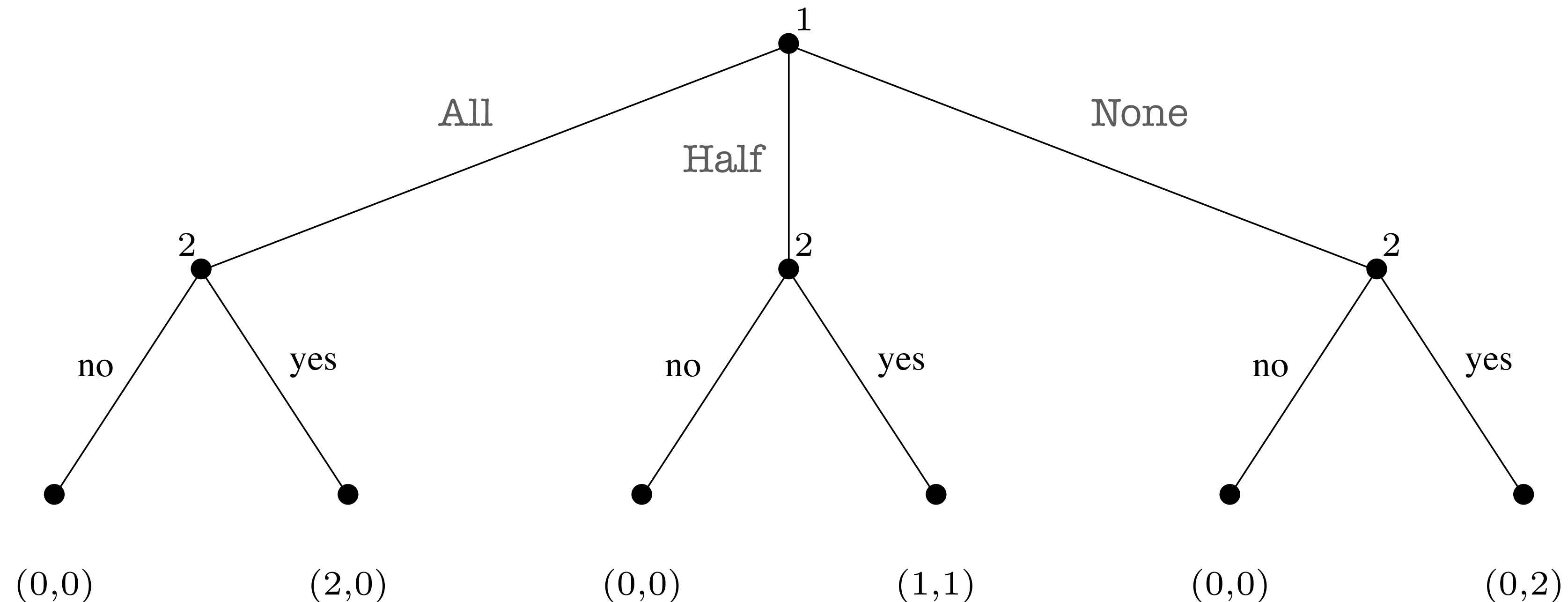
Definition:
A finite perfect-information game in extensive form is a tuple
$G = (N, A, H, Z, \chi, \rho, \sigma, u)$, where

- N is a set of n **players**,

- A is a single set of **actions**,

- H is a set of nonterminal **choice nodes**,

- Z is a set of **terminal nodes** (disjoint from H),

- $\chi : H \to 2^A$ is the **action function**,

- $\rho : H \to N$ is the **player function**,

- $\sigma : H \times A \to H \cup Z$ is the **successor function**,

- $u = (u_1, u_2, \ldots, u_n)$ is a **utility function** for each player, $u_i : Z \to \mathbb{R}$

Fun Game:
The Sharing Game

- Two siblings must decide how to share two $100 coins

- Sibling 1 suggests a division, then sibling 2 **accepts** or **rejects**

  - If **rejected**, nobody gets any coins.

Pure Strategies

Question: What are the **pure strategies** in an extensive form game?

Definition:
Let $G = (N, A, H, Z, \chi, \rho, \sigma, u)$ be a perfect information game in extensive form. Then the **pure strategies of player** i consist of the cross product of actions available to player i at each of their choice nodes, i.e.,

$$\prod_{h \in H | \rho(h) = i} \chi(h)$$

- A pure strategy associates an action with **each** choice node, even those that will **never be reached**

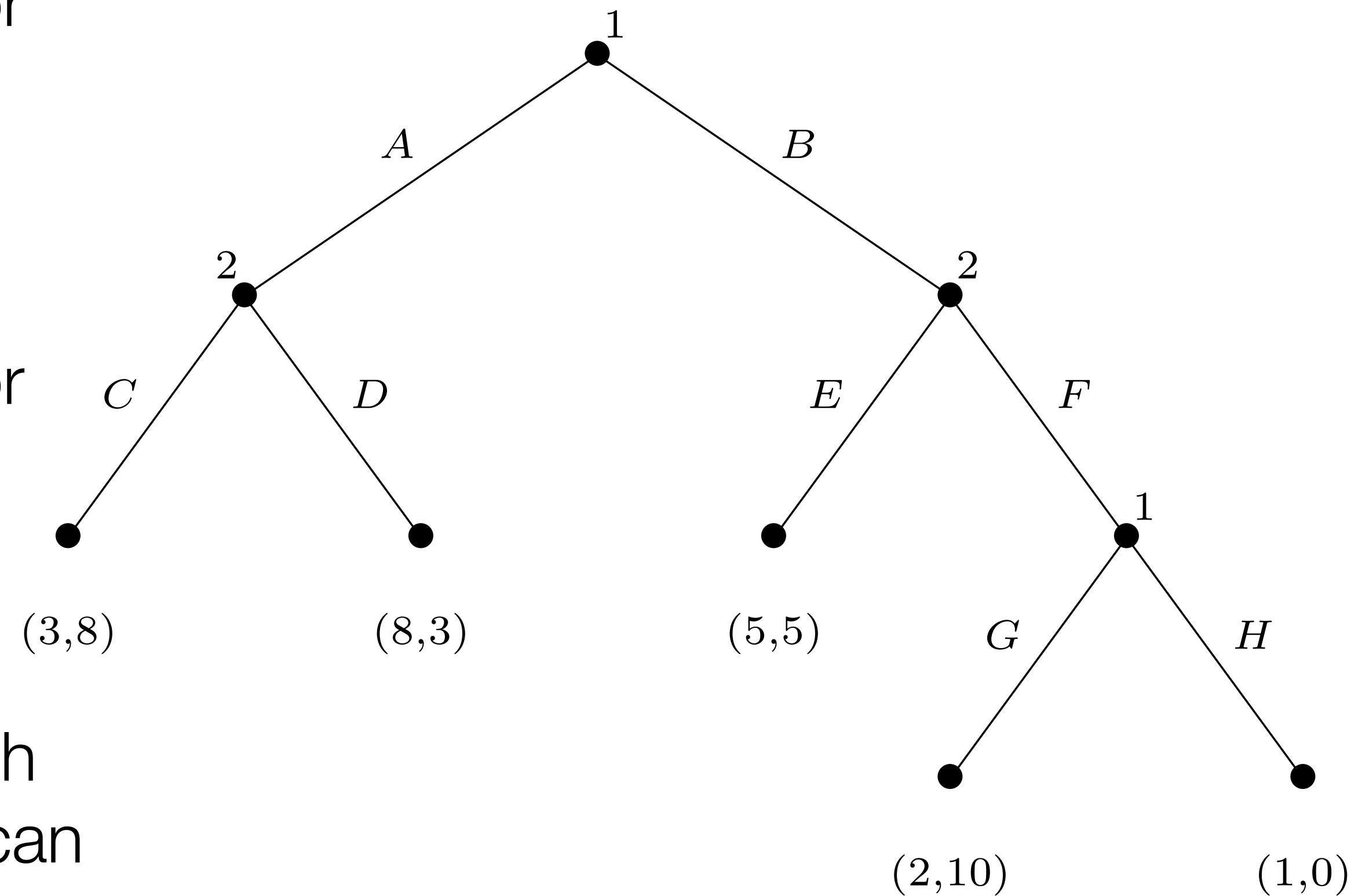 - Even nodes that will never be reached as a result of the strategy itself!

Pure Strategies Example

Question: What are the **pure strategies** for **player 2**?

- $\{(C,E),(C,F),(D,E),(D,F)\}$

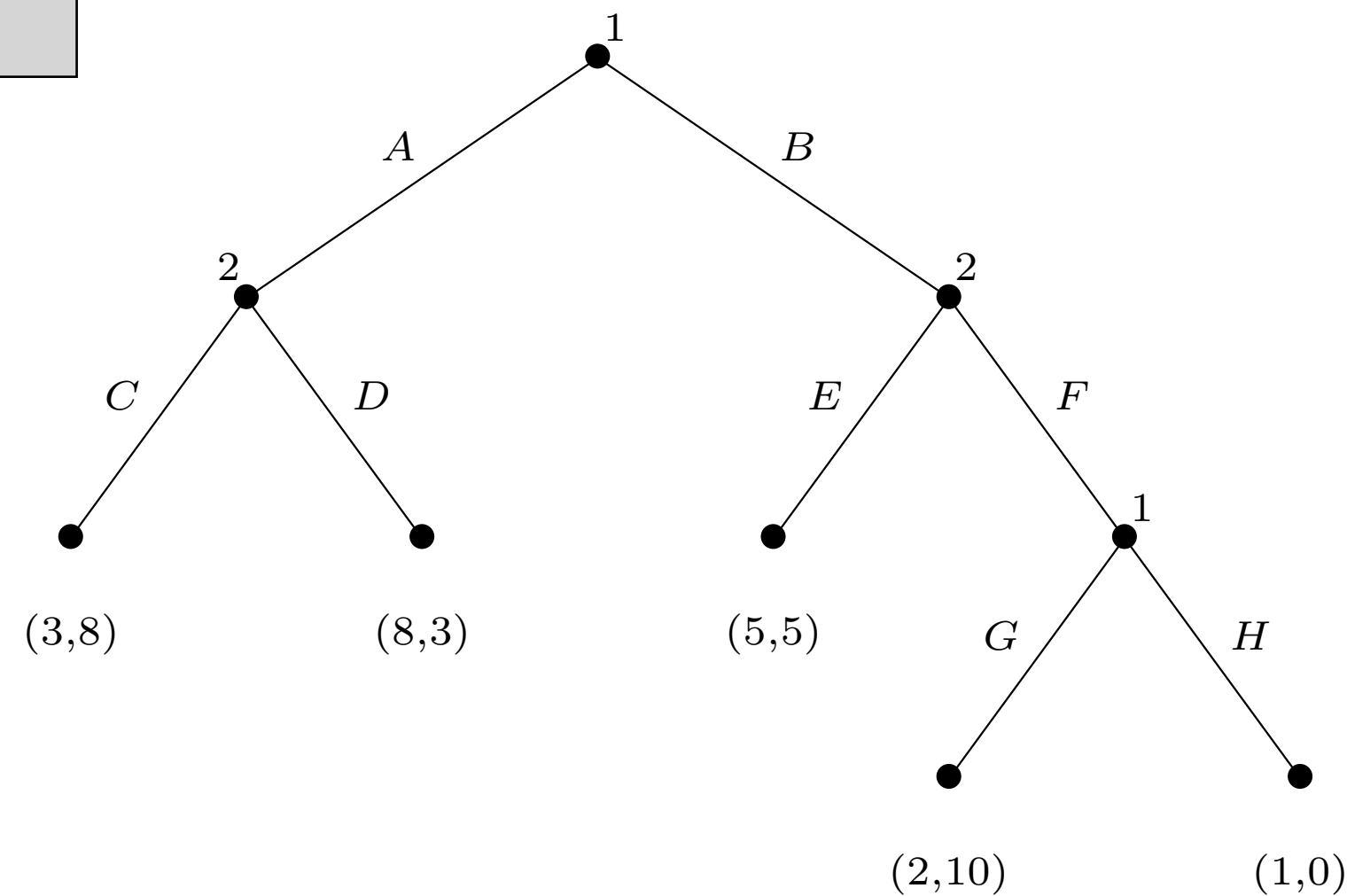**Question:** What are the **pure strategies** for **player 1**?

- $\{(A,G),(A,H),(B,G),(B,H)\}$

- Note that these associate an action with the second choice node even when it can never be reached

Induced Normal Form

	C,E	C,F	D,E	D,F
A,G	3,8	3,8	8,3	8,3
A,H	3,8	3,8	8,3	8,3
B,G	5,5	2,10	5,5	2,10
B,H	5,5	1,0	5,5	1,0

- Any pair of pure strategies uniquely identifies a **terminal node**, which identifies a **utility** for each agent (**why?**)

- We have now defined a set of **agents**, **pure strategies**, and **utility functions**

- Any extensive form game defines a corresponding **induced normal form game**

Reusing Old Definitions

- We can plug our new definition of **pure strategy** into our existing definitions for:

    - Mixed strategy

    - Best response

    - Nash equilibrium (both pure and mixed strategy)

Question:

What is the definition of a **mixed strategy** in an extensive form game?

Pure Strategy Nash Equilibria

Theorem: [Zermelo, 1913]
Every finite perfect-information game in extensive form has at least one **pure strategy Nash equilibrium**.

- Starting from the bottom of the tree, no agent needs to **randomize**, because they already know the best response

- There might be **multiple** pure strategy Nash equilibria in cases where an agent has multiple best responses at a **single choice node**
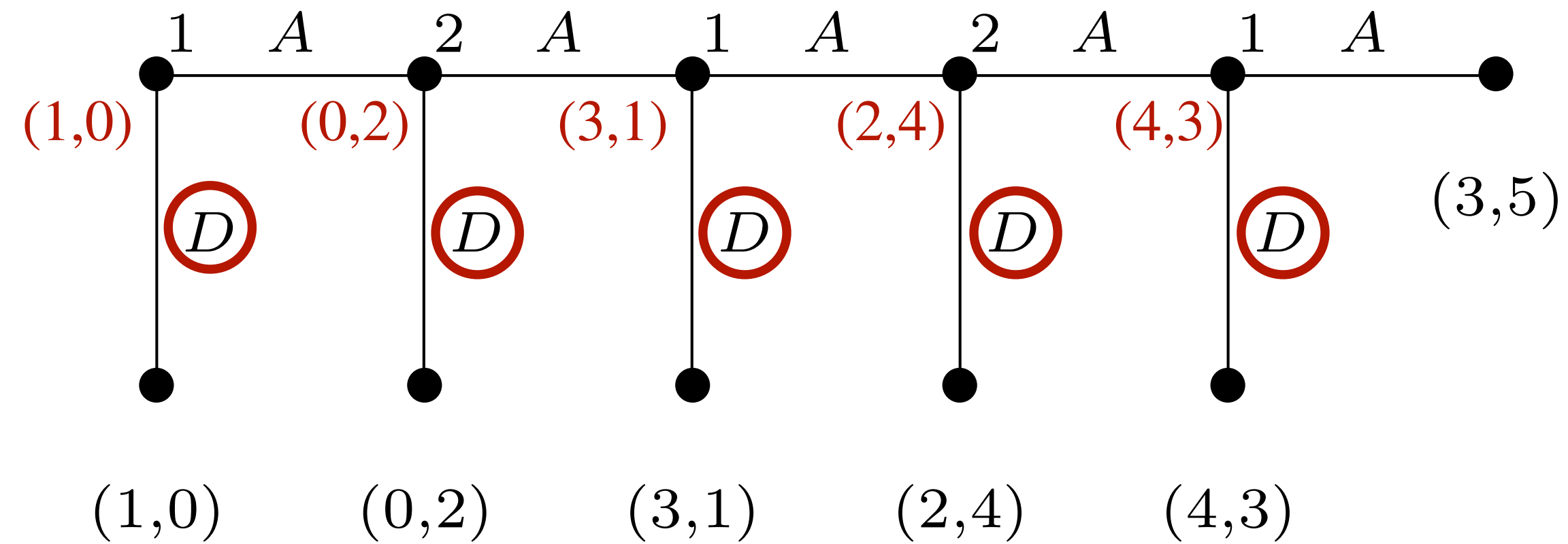
Backward Induction

- **Backward induction** is a straightforward algorithm that is guaranteed to compute a **pure strategy Nash equilibrium**.

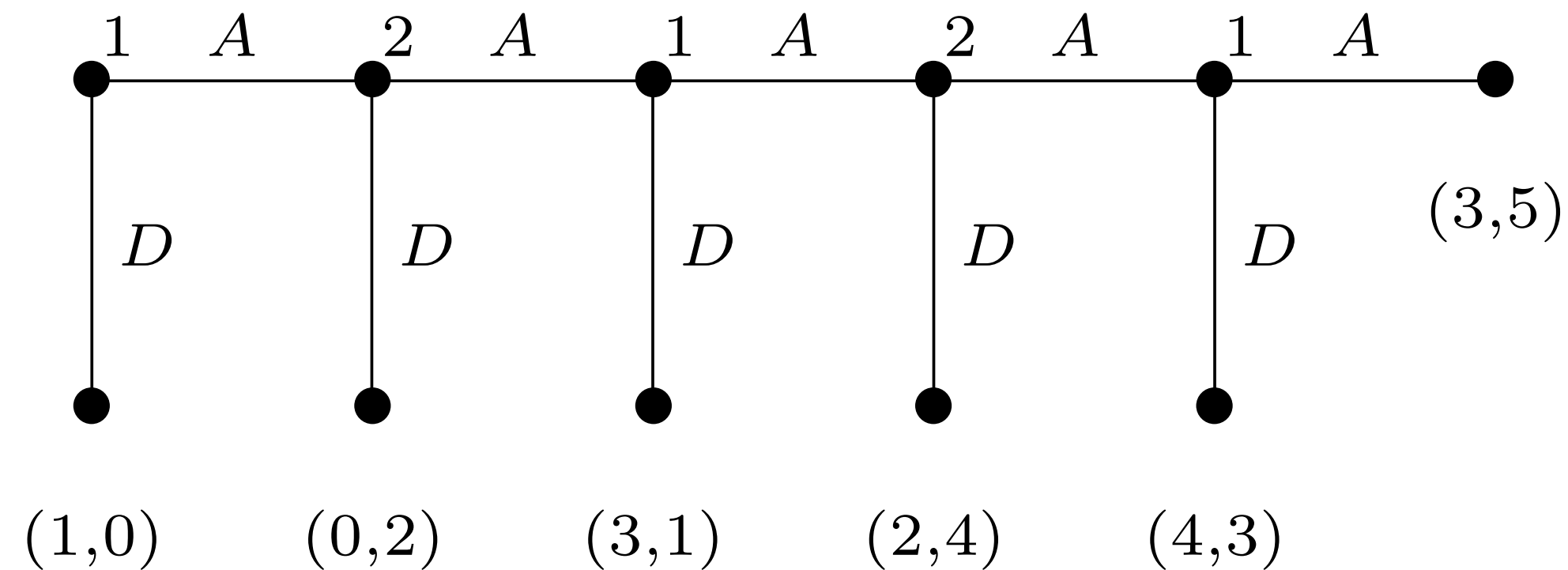- **Idea:** Replace subgames lower in the tree with their equilibrium values

$$s_1^* = (A, G)$$
$$s_2^* = (C, F)$$
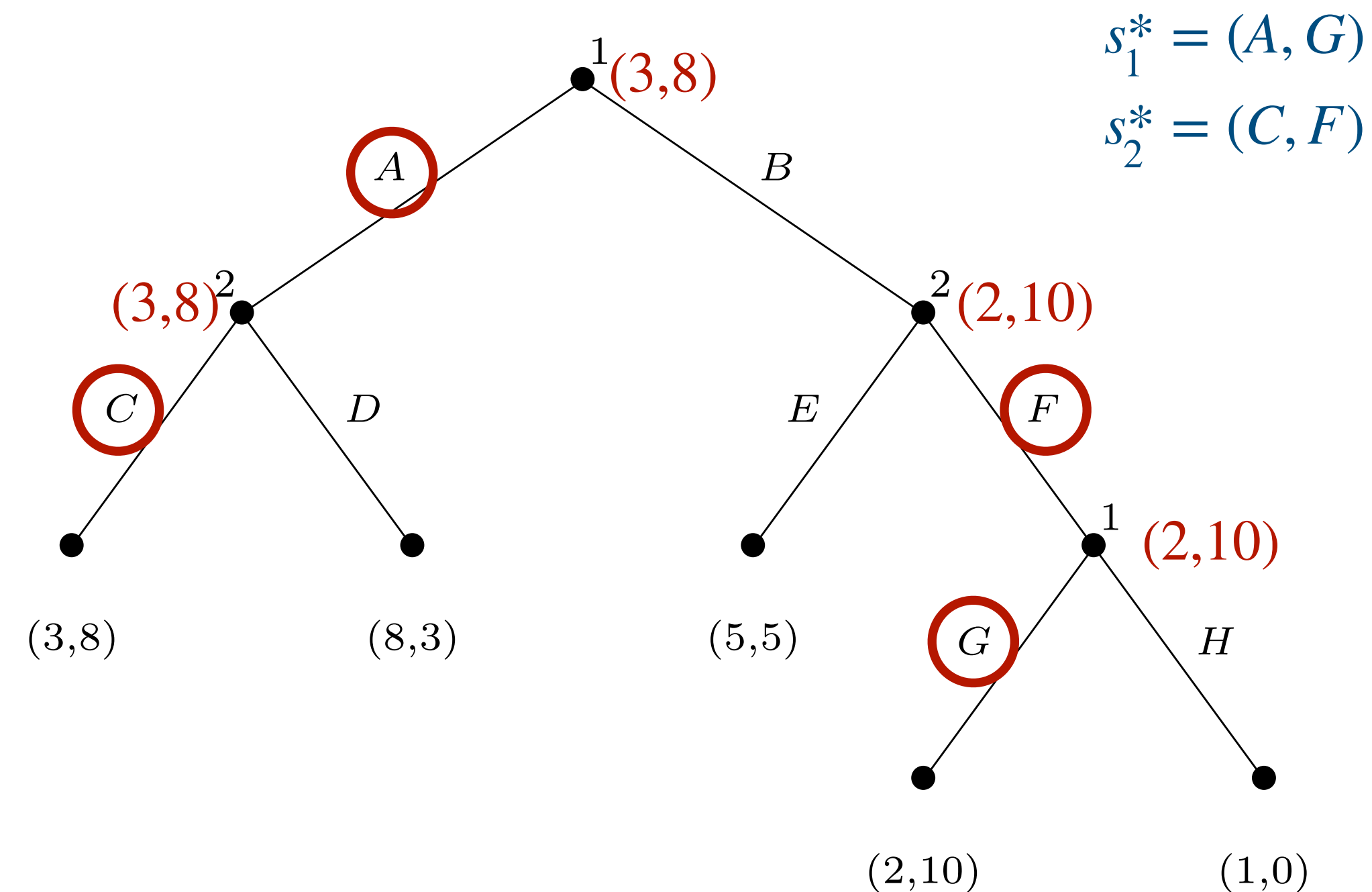
Question:

Could you use backward induction to compute a **maxmin** strategy? How?

Fun Game: Centipede

1 A	2 A	1 A	2 A	1 A
(1,0)	(0,2)	(3,1)	(2,4)	(4,3)
D	D	D	D	D

(3,5)

(1,0)   (0,2)   (3,1)   (2,4)   (4,3)

- At each stage, one of the players can go **Across** or **Down**

- If they go **Down**, the game ends.

Backward Induction Criticism

- The **unique** equilibrium is for each player to play **Down at the first opportunity.**

- **Empirically**, this is not how real people tend to play!

- **Theoretically**, what should you do if you arrive at an **off-path** node?

  - How do you update your beliefs to account for this probability 0 event?

  - If player 1 knows that you will update your beliefs in a way that causes you not to play **Down**, then playing **Down** is no longer their only rational choice...

Summary

- **Mixed strategies** are distributions over pure strategies

  - In normal form games, pure strategies are just single actions

- **Extensive form games** model sequential actions

- **Pure strategies** for extensive form games map choice nodes to actions

  - Induced normal form: normal form game with these pure strategies

  - Notions of mixed strategy, best response, etc. translate directly

- **Perfect information:** Every agent sees all actions of the other players

  - Backward induction computes a pure strategy Nash equilibrium for any perfect information extensive form game