Solving Go

CMPUT 355: Games, Puzzles, and Algorithms

Solving 2x2 Go

CMPUT 355: Games, Puzzles, and Algorithms

Lecture Outline

1. Logistics & Recap
2. Nim, part 2½
3. 2x2 Go

Logistics

- **Practice questions #3** are available
    - ⚠️ **Corrected solutions released this morning** ⚠️
- **Quiz 3** is **this Friday** (Feb 27)
    - *Coverage:* up to and including **Feb 13** (Move Ordering & Nim)
    - Bring your student ID!
    - No calculators or other devices
- **TA Office hours:** every Thursday 1pm-2pm in **UCOMM-3-136**

Recap: XorSum

- **xorsum** operation: Cumulative bitwise XOR of **each binary digit** of the pile sizes
- Examples:
  - xorsum(1,2,3) = 00
    - $\begin{array}{l} 1 = 01 \\ 2 = 10 \\ 3 = 11 \end{array}$ and $\begin{array}{l} 0 \oplus 1 \oplus 1 = 0 \\ 1 \oplus 0 \oplus 1 = 0 \end{array}$
  - xorsum(3,5,3) = 101b = 5
    - $\begin{array}{l} 3 = 011 \\ 5 = 101 \\ 3 = 011 \end{array}$ and $\begin{array}{l} 0 \oplus 1 \oplus 0 = 1 \\ 1 \oplus 0 \oplus 1 = 0 \\ 1 \oplus 1 \oplus 1 = 1 \end{array}$

Recap: Nim Formula Theorem

Theorem: A Nim position with pile sizes $p_1, \ldots, p_k$ is **losing** iff

$$\text{xorsum}(p_1, \ldots, p_k) = 0.$$

Proof sketch:

1. If $\text{xorsum(P)} = 0$, then $\text{xorsum}(C) \neq 0$ for **every** child C of P
2. If $\text{xorsum}(P) \neq 0$, then $\text{xorsum}(C) = 0$ for **some** child C of P
3. $\text{xorsum}(0,0,\ldots,0) = 0$
4. Any position that has $(0,0,\ldots,0)$ as a child is a **winning** position
5. Complete the proof by induction

Nim Formula Theorem Proof Step 2

2. If $\mathrm{xorsum}(P) \neq 0$, then $\mathrm{xorsum}(C) = 0$ for **some** child C of P

1	0	0	1	0
1	1	0	1	0
1	0	1	1	1
0	1	1	0	0
1	1	0	1	1
0	0	0	1	1
0	1	0	1	1
4	3	2	1	0

Nim Formula Theorem Proof Step 2

2. If $\operatorname{xorsum}(P) \neq 0$, then $\operatorname{xorsum}(C) = 0$ for **some** child C of P

(a) Let d be the digit number of the most significant 1 in the binary representation of $\operatorname{xorsum}(P)$

1	0	0	1	0
1	1	0	1	0
1	0	1	1	1
0	1	1	0	0
1	1	0	1	1
0	0	0	1	1
0	1	0	1	1
4	3	2	1	0

$d = 3$

Nim Formula Theorem Proof Step 2

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(a) Let d be the digit number of the most significant 1 in the binary representation of $\mathrm{xorsum}(P)$

(b) There exists at least one j such that p_j has a 1 in digit d (**why?**)

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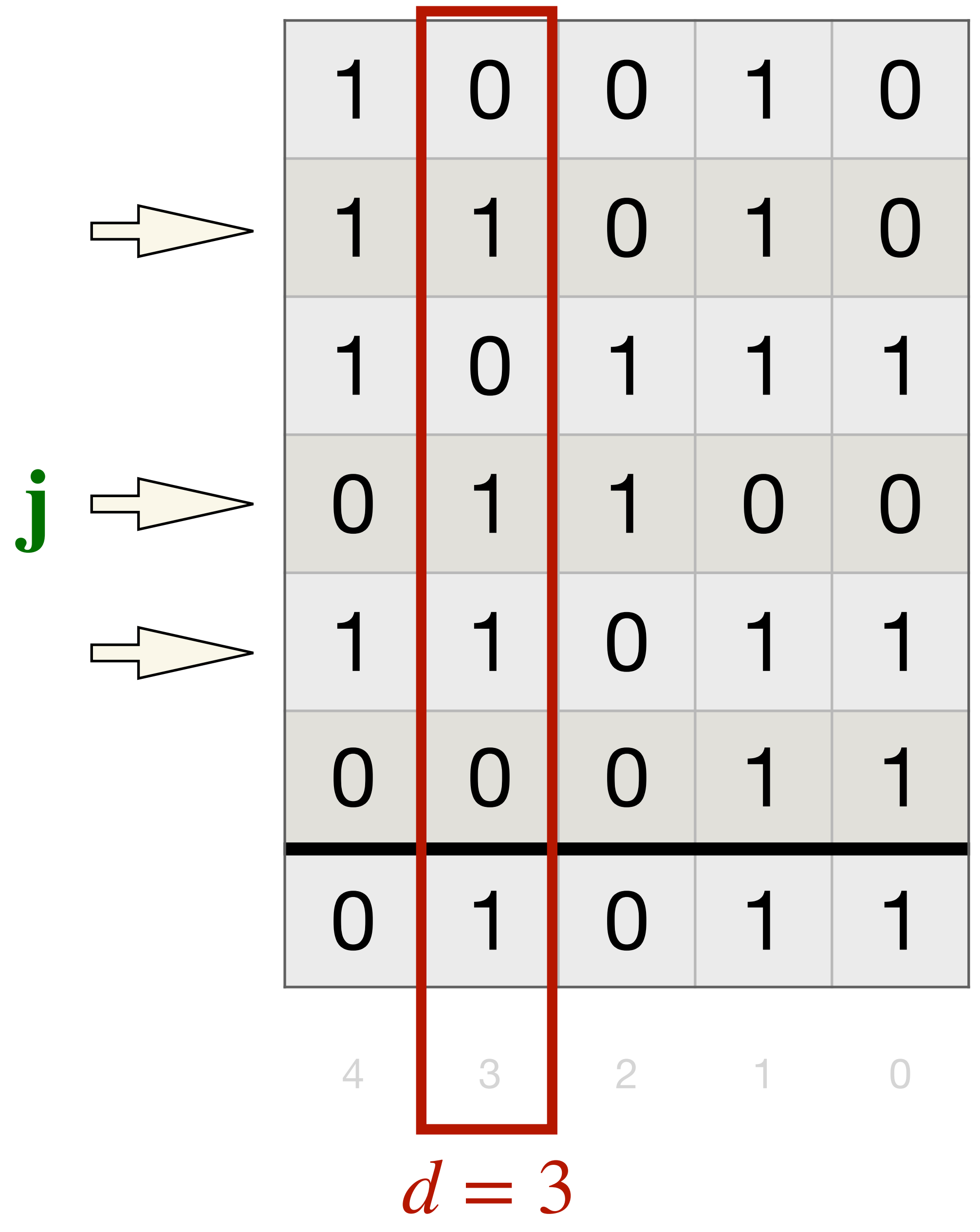

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(c) $\operatorname{xorsum}(p_1, \ldots, p_{j-1}, p_{j+1}, \ldots, p_k)$ has a 0 in the d-th column (**why?**)

- and *also* in every more-significant column that p_j has a 0 in (**why?**)

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	1	0	1	1	1
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(d) So $\mathrm{xorsum}(p_1, \ldots, p_{j-1}, p_{j+1}, \ldots, p_k) < p_j$ (**why?**)

	4	3	2	1	0
	1	0	0	1	0
j	1	1	0	1	0
	1	0	1	1	1
	0	1	1	0	0
	1	1	0	1	1
	0	0	0	1	1
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4	3	2	1	0
1	0	0	1	0
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(e) Set $p'_j = \operatorname{xorsum}(p_1, \ldots, p_{j-1}, p_{j+1}, \ldots, p_k)$

	1	0	0	1	0
j	1	1	0	1	0
	1	0	1	1	1
	0	1	1	0	0
	1	1	0	1	1
	0	0	0	1	1
	0	1	0	1	1
	4	3	2	1	0
		$d = 3$			

1	0	0	1	0
1	0	1	1	1
0	1	1	0	0
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(e) Set $p_j' = \mathrm{xorsum}(p_1, \ldots, p_{j-1}, p_{j+1}, \ldots, p_k)$

	1	0	0	1	0
j	1	1	0	1	0
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	0	1	1	0	0
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	0	0	0	1	1
	0	1	0	1	1
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$d = 3$

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(f)

$$\begin{aligned} \mathrm{xorsum}(p_1, \ldots, p_{j-1}, \mathbf{p_j'}, p_{j+1}, \ldots, p_k) &= \mathrm{xorsum}(p_1, \ldots, p_{j-1}, p_{j+1}, \ldots, p_k) \oplus p_j' \\ &= p_j' \oplus p_j' \\ &= 0 \end{aligned}$$

	1	0	0	1	0
j	1	1	0	1	0
	1	0	1	1	1
	0	1	1	0	0
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&= p_j' \oplus p_j' \\
&= 0
\end{aligned}$$

■

	1	0	0	1	0
j	1	1	0	1	0
	1	0	1	1	1
	0	1	1	0	0
	1	1	0	1	1
	0	0	0	1	1
	0	1	0	1	1
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$d = 3$

1	0	0	1	0
1	0	0	0	1
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0	1	1	0	0
1	1	0	1	1
0	0	0	1	1
0	**0**	**0**	**0**	**0**
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$d = 3$

Nim Formula Theorem Proof Step 5

1. If $\text{xorsum}(\text{P}) = 0$, then $\text{xorsum}(C) \neq 0$ for **every** child C of P
2. If $\text{xorsum}(P) \neq 0$, then $\text{xorsum}(C) = 0$ for **some** child C of P
3. $\text{xorsum}(0,0,\ldots,0) = 0$
4. Any position that has $(0,0,\ldots,0)$ as a child is a **winning** position

Nim Formula Theorem Proof Step 5

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$(\mathrm{xorsum}(P) \neq 0 \implies P$ is winning) and $(\mathrm{xorsum}(P) = 0 \implies P$ is losing)

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Then for all P that are within $k+1$ steps of $(0,\ldots,0)$:

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$(\mathrm{xorsum}(P) \neq 0 \implies P$ is winning) and $(\mathrm{xorsum}(P) = 0 \implies P$ is losing)

Then for all P that are within $k+1$ steps of $(0,\ldots,0)$:

(a) $\mathrm{xorsum}(P) \neq 0$:

- by (2), $\mathrm{xorsum}(C) = 0$ for **some** child C
- C is within k of $(0,\ldots,0)$, so by IH C is **losing**
- So P has **at least one losing child**, and is therefore **winning**

Nim Formula Theorem Proof Step 5

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(b) $\mathrm{xorsum}(P) = 0$:

- by (1), $\mathrm{xorsum}(C) \neq 0$ for **all** children C
- every C is within k of $(0,\ldots,0)$, so by IH every C is **winning**
- So **all of P's children are winning**, hence P is **losing**

Nim Formula Theorem Proof Step 5

1. If $\text{xorsum}(\text{P}) = 0$, then $\text{xorsum}(C) \neq 0$ for **every** child C of P
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Inductive hypothesis: suppose that for all positions P within k steps of $(0,\ldots,0)$, $(\text{xorsum}(P) \neq 0 \implies P$ is winning$)$ and $(\text{xorsum}(P) = 0 \implies P$ is losing$)$

Then for all P that are within $k+1$ steps of $(0,\ldots,0)$:

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- by (1), $\text{xorsum}(C) \neq 0$ for **all** children C
- every C is within k of $(0,\ldots,0)$, so by IH every C is **winning**
- So **all of P's children are winning**, hence P is **losing**

Base case:

(a) By (3), every position P within 0 of $(0,\ldots,0)$ has $\text{xorsum}(C) = 0$, and by definition it is a losing position

(b) By (1), every position P within 1 of $(0,\ldots,0)$ has $\text{xorsum}(C) \neq 0$, and by (4) they are all winning $\blacksquare$

Recap: Using the Nim Formula

- **Checking** if a position P is **winning** or **losing**: Simply compare $\mathrm{xorsum}(P)$ to 0
- Finding a **winning move**:
  1. Compute $\mathrm{xorsum}(P)$
  2. Find d and a pile j with a 1 in digit d
     - Equivalently: a pile j with $p_j \geq \mathrm{xorsum}(p_1, \ldots, p_{j-1}, p_{j+1}, \ldots, p_k)$
  3. Set $p_j' = \mathrm{xorsum}(p_1, \ldots, p_{j-1}, p_{j+1}, \ldots, p_k)$
  4. Take $p_j - p_j'$ stones from pile j
     - i.e., move to position $(p_1, \ldots, p_{j-1}, \mathbf{p_j'}, p_{j+1}, \ldots, p_k)$

Nim Formula Implementation: **nim/nim.py**

```
def xorsum(L):
  xsum = 0
  for j in L:
      xsum ^= j
  return xsum
```

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```
def nimreport(P): # report all winning nim moves from P, use formula
  total = xorsum(P)
  if total==0:
    print(' loss')
    return
  for j in P:
    tj = total^j
    if j >= tj:
      print(' win: take',j - tj,'from pile with',j)
```

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  for j in P:
    tj = total^j
    if j >= tj:
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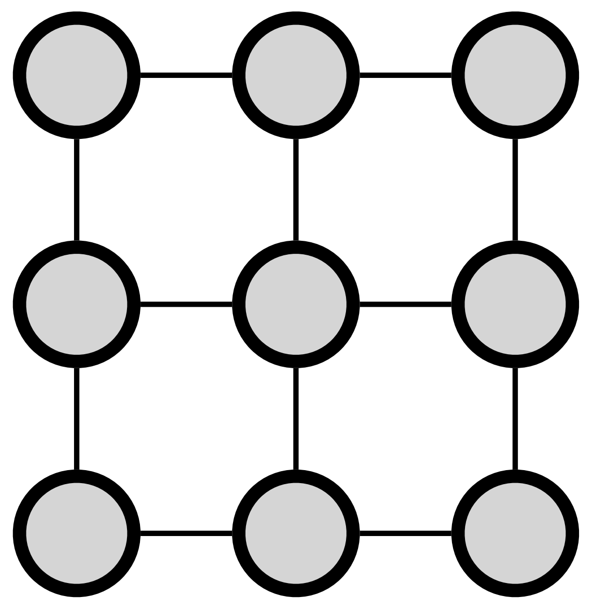
```

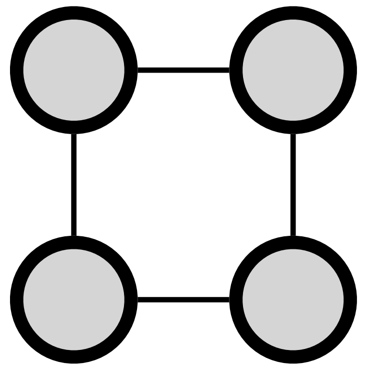
Note: $\mathrm{xorsum}(p_1, \ldots, p_{j-1}, p_{j+1}, \ldots, p_k) = \mathrm{xorsum}(p_1, \ldots, p_k) \oplus p_j$ **(why?)**

Recap: Go!



1. Played on a grid of **points** between players Black and White
2. Each point is either Black, White, or Empty
3. A **move** consists of either Pass, or colouring one point (placing a stone)
4. An Empty point adjacent to a point is a **liberty** of that point
5. Connected **groups** of the same colour share liberties
6. After placing a stone, any group of the other colour with no liberties is **captured**
7. **Suicide:** After all captures, the placed stone must belong to a group with at least 1 liberty
8. **Positional superko:** After a move, the position must differ from all previous positions

2×2 Go: Search Graph



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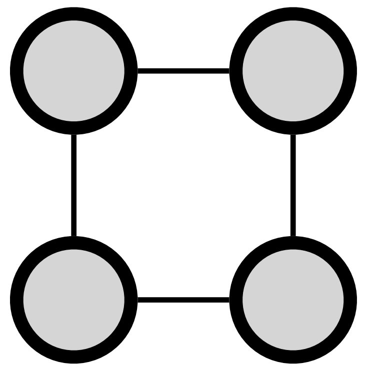

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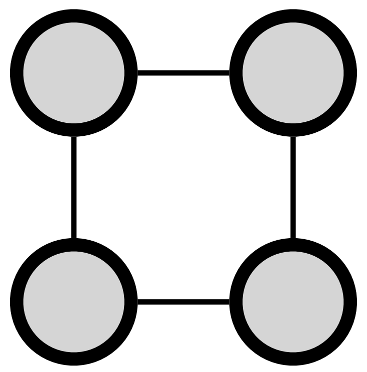

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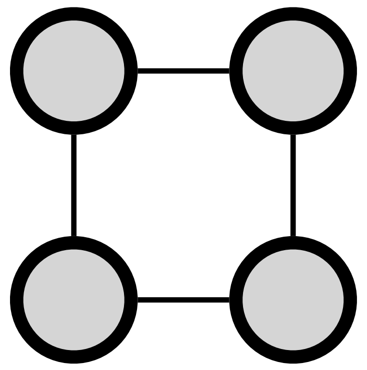

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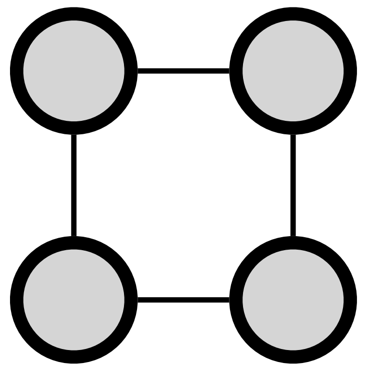

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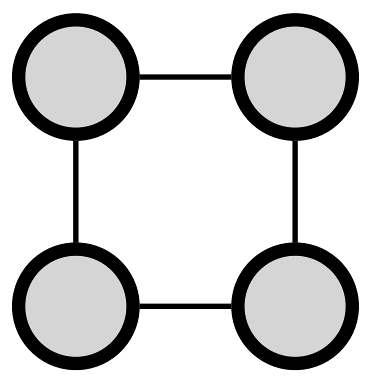

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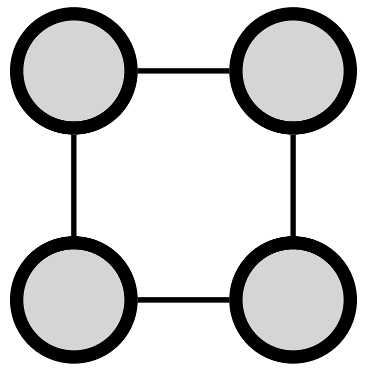

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- It turns out there are **386,356,909,593** legal games of 2×2 Go
  - For 3×3 Go, the number is approximately 10^{1100}

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- **Question:** Which move ordering will yield the most pruning?

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  3. **Alpha-beta** pruning with **pass moves first**:
     Searches $1{,}446$ **nodes** (with a maximum depth of **22**)
- **Pass-first** ordering yields dramatic improvements in pruning:
  - Gets to the closest possible terminal
  - Establishes alpha/beta values very quickly

Minimax Proof for 2×2 Go

- It turns out that the minimax value of 2×2 Go is **1**
- **Question: How many proof trees** do we need to prove that? (**why?**)

Source: https://webdocs.cs.ualberta.ca/~hayward/355/ssgo.pdf

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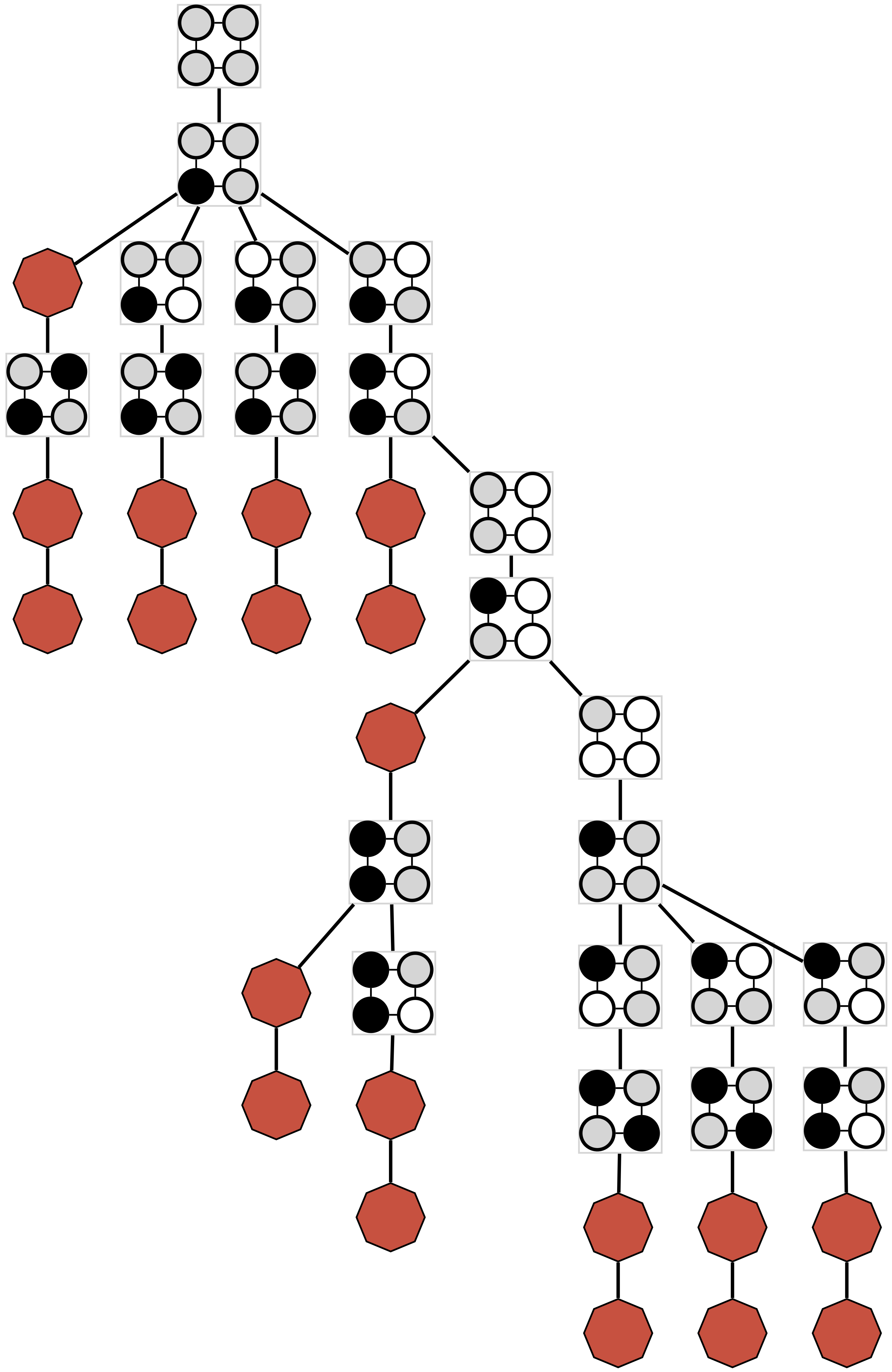

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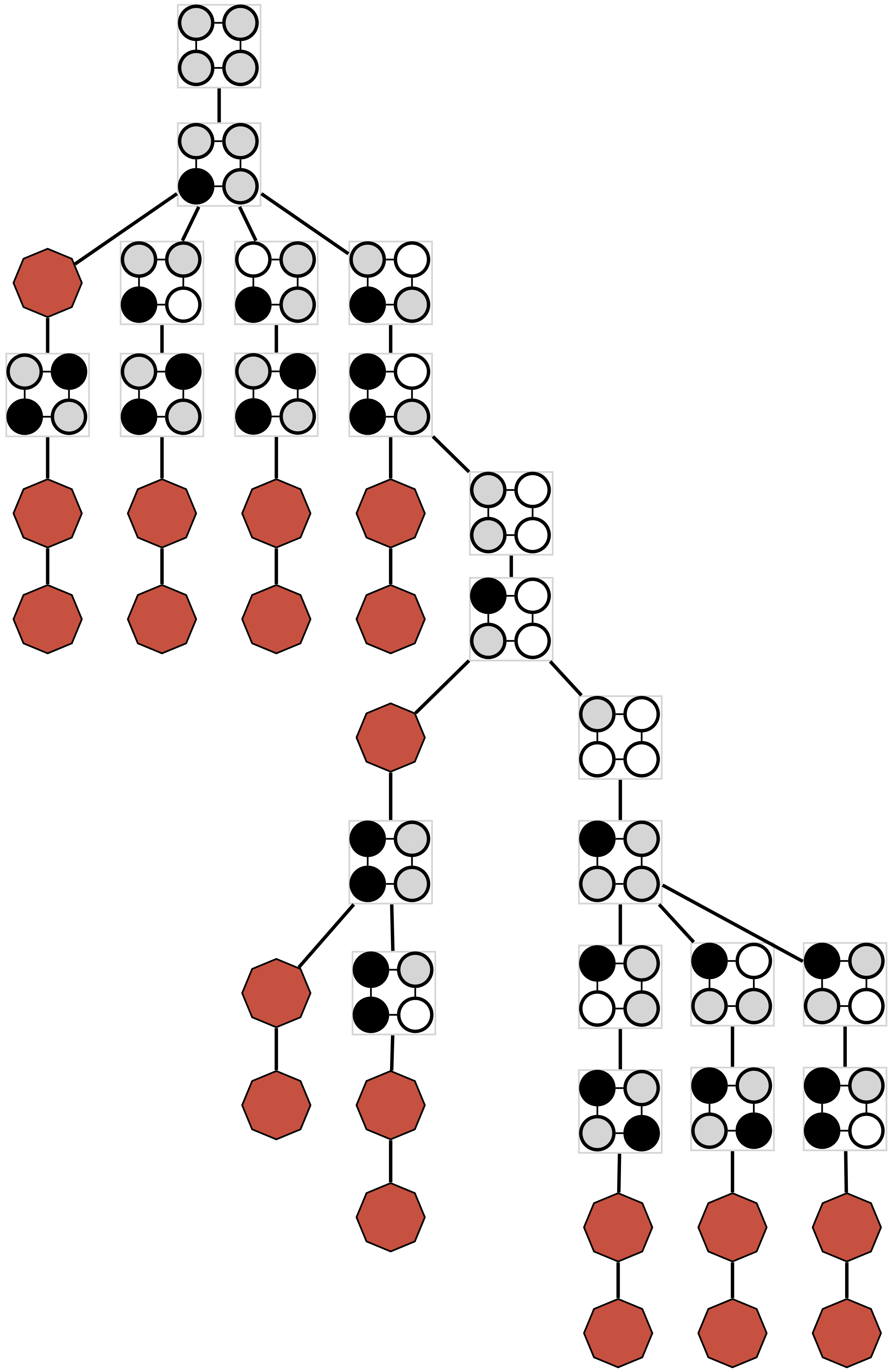

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Summary

- **Nim formula:**
  - Tells us whether a position is **winning** or **losing without search**
  - Allows us to compute a **winning move** from a winning position
    - Without search
    - Without even looking at **every move!**
- **2x2 Go:**
  - Can be solved using **just alpha-beta search**
    - Without alpha-beta: hopeless
  - **Pass-first** move ordering dramatically reduces the search space
- 3x3 Go is **significantly harder**
  - So 2x2 Go is right at the edge of feasibility for alpha-beta-based techniques