Nim, part 2

CMPUT 355: Games, Puzzles, and Algorithms

Lecture Outline

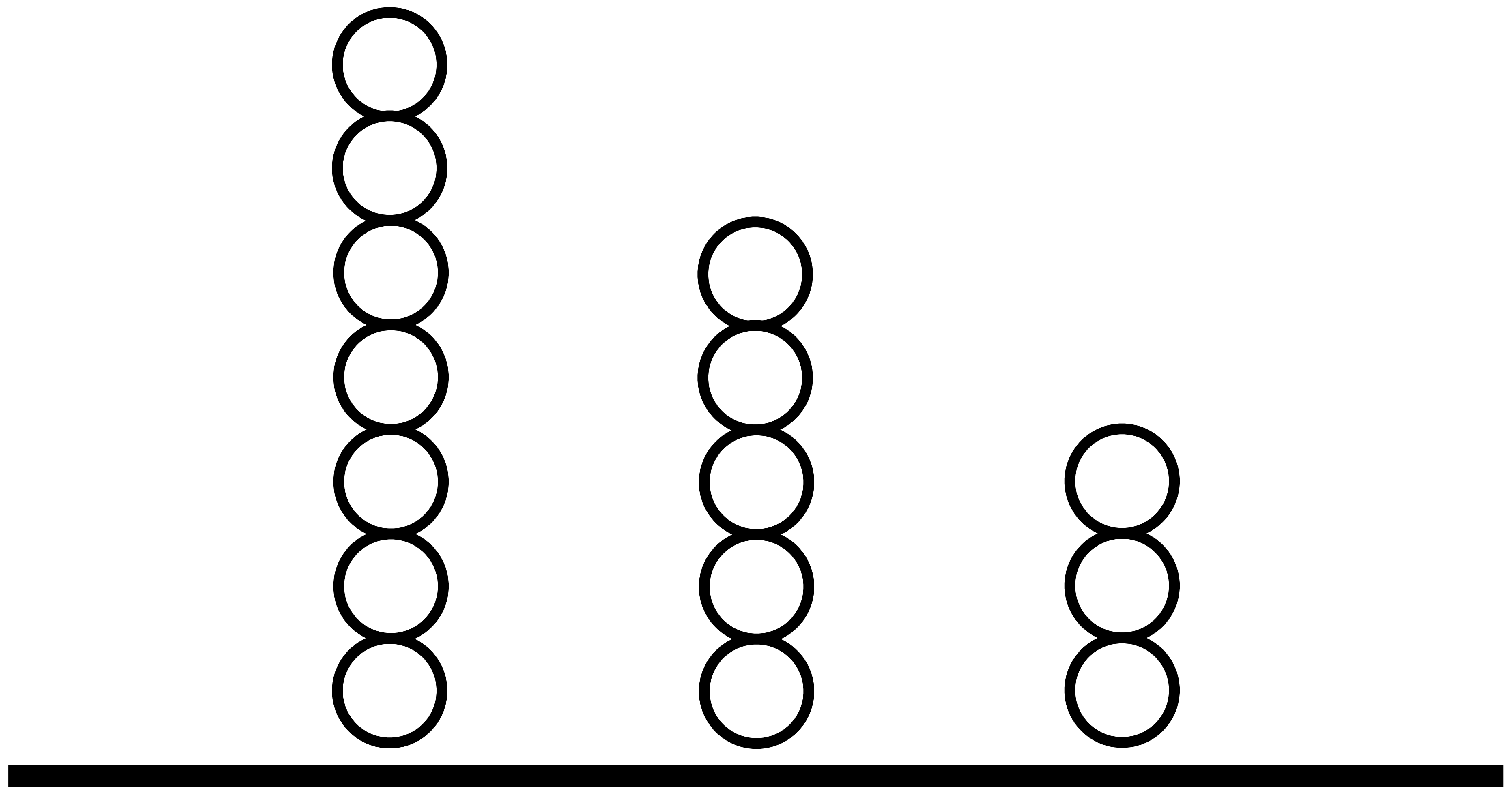
1. Logistics & Recap
2. Dynamic programming
3. Nim Formula & Theorem

Logistics

- **Quiz 2** marks are **released** on Canvas
  - A scan of your quiz is attached to the Canvas assignment
- **Practice questions #3** are available
  - Solutions released last Friday
- **Quiz 3** is **this Friday** (Feb 27)
  - *Coverage:* up to and including **Feb 13** (Move Ordering & Nim)
  - Bring your student ID!
  - No calculators or other devices
- **TA Office hours:** every Thursday 1pm-2pm in **UCOMM-3-136**

Recap: Nim

- There are k piles of stones
- A move is removing some **positive** number of stones from a **single pile**
- Players alternate moves
- Last player to move wins
  - i.e., if there are **no stones left** and you are player-to-move, you lose



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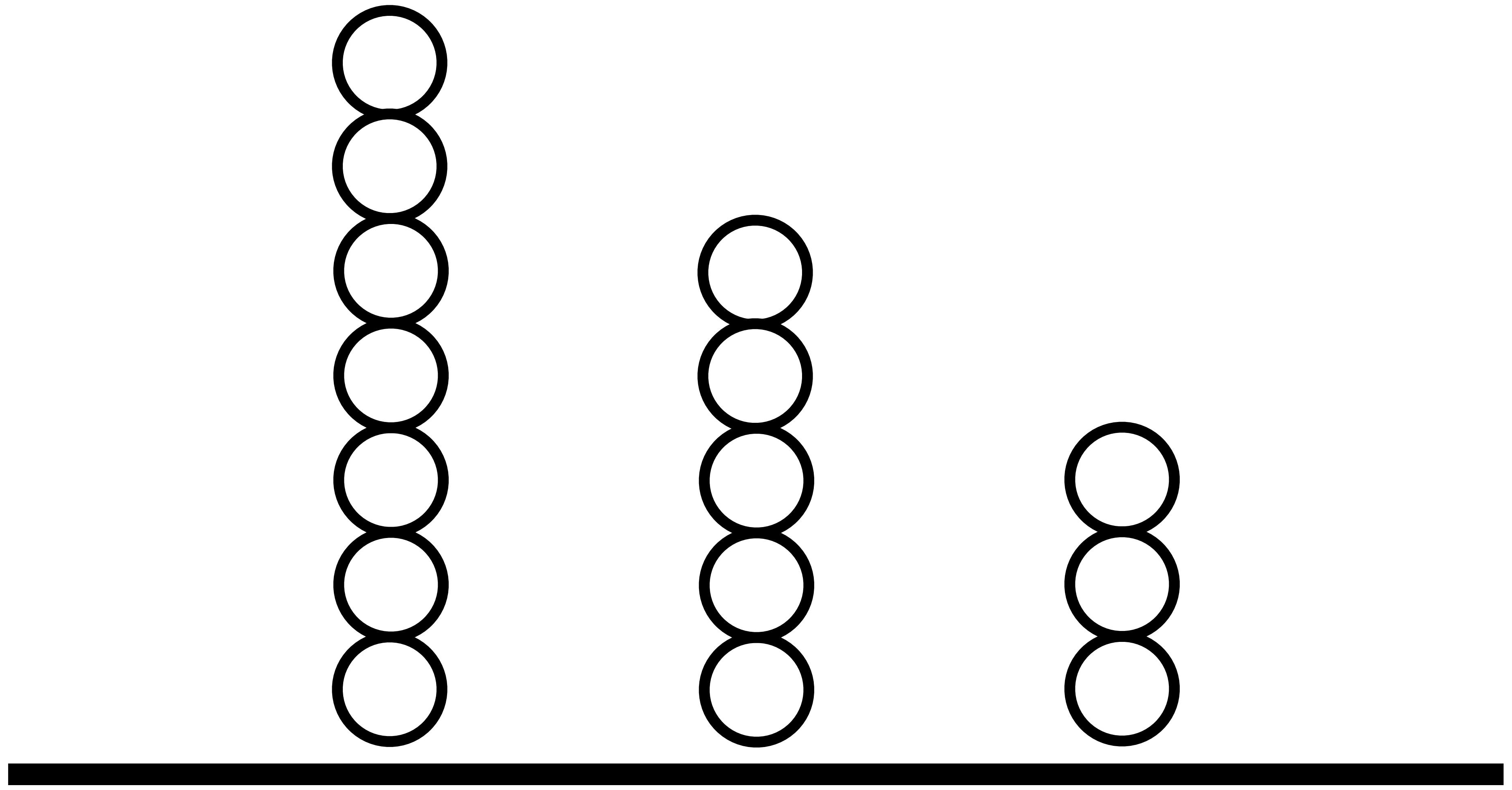

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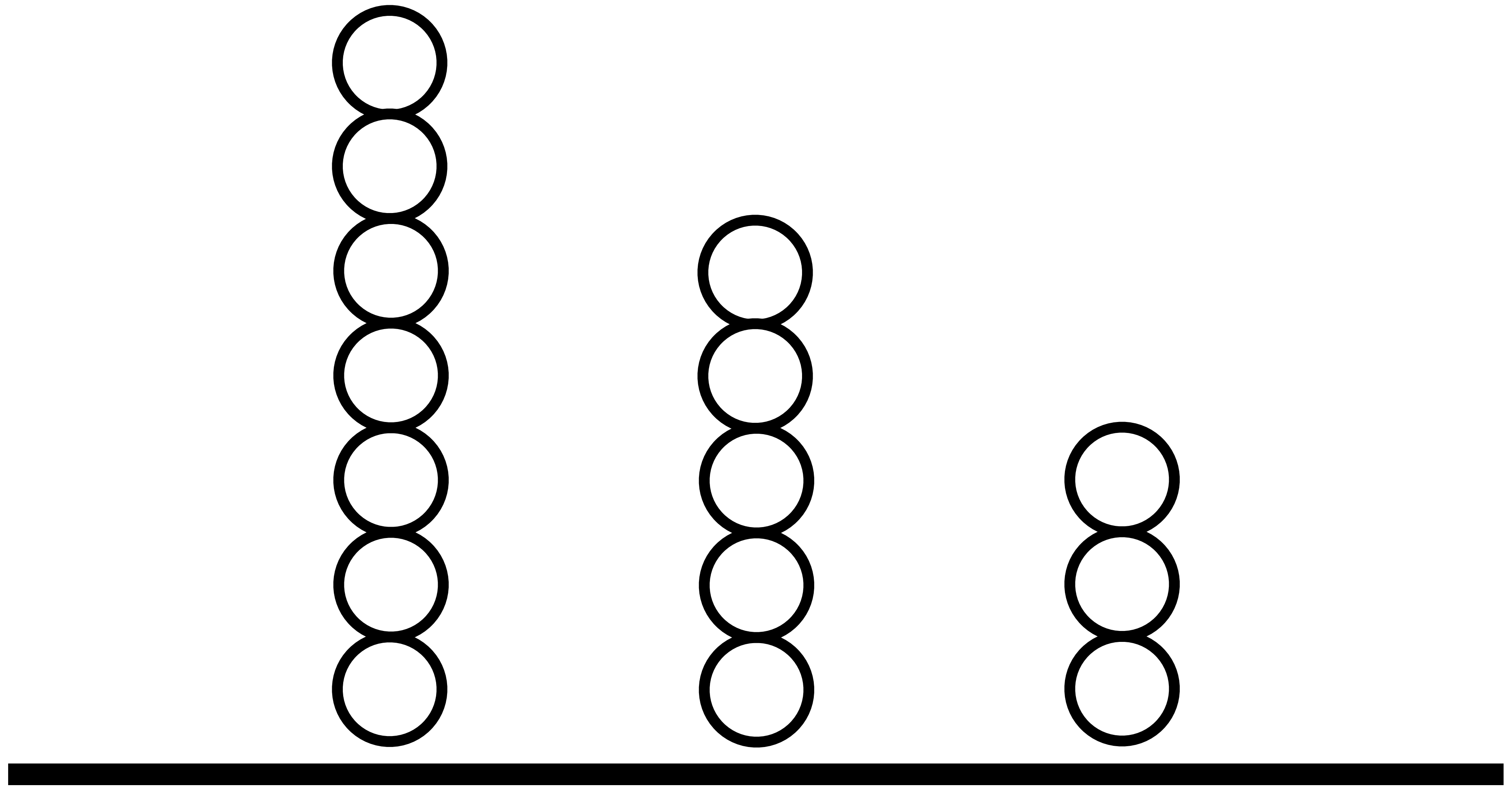
- Example at right is position $7{,}5{,}3$
- Every position is either **winning** (i.e., player-to-move can force a win from here) or **losing** (every child position is a winning position for the opponent)



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```
def fib(n, D):
  if n in D:
    return D[n]
  elif n <= 1:
      return 1
  else:
    f = fib(n-1, D) + fib(n-2, D)
    D[n] = f
    return f
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- **Question:** What would that look like in the context of Nim?

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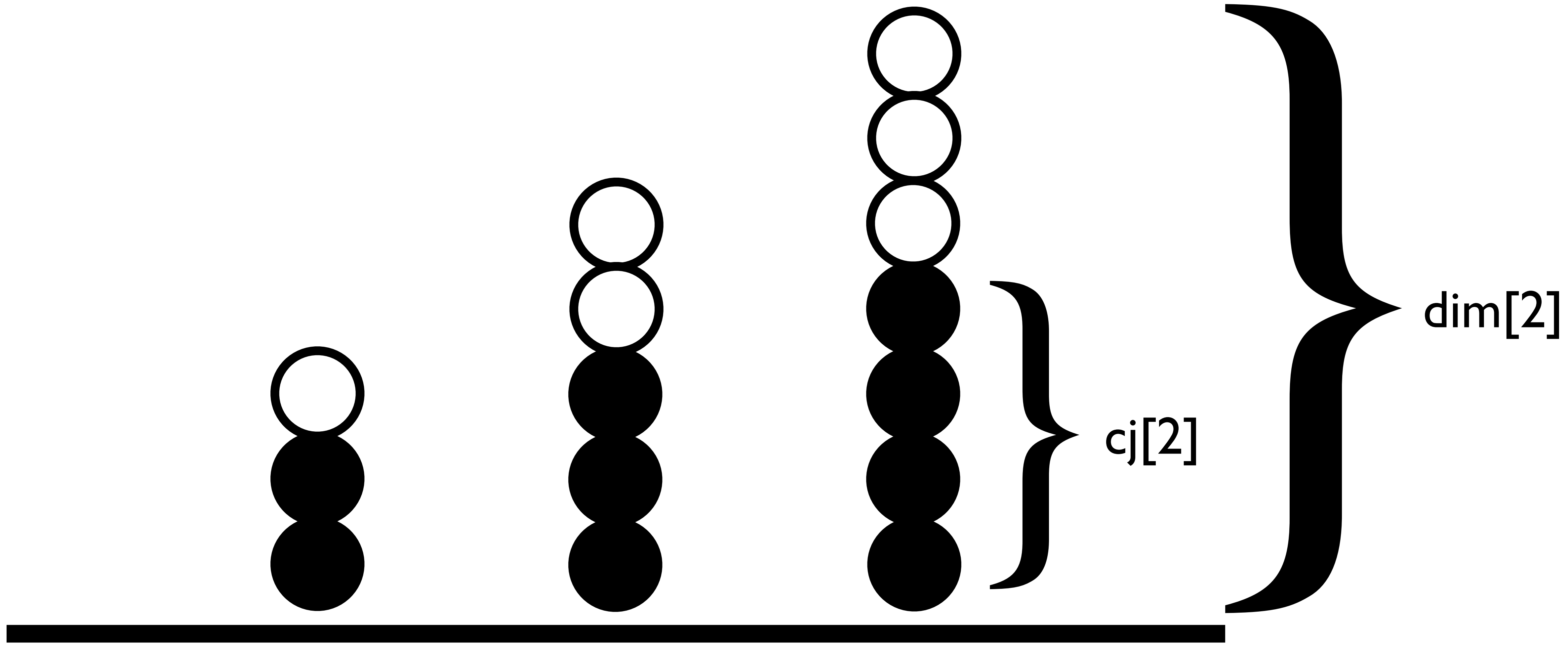
- **Claim:** every child position will have a **smaller number** than its parent (**why?**)
- So if we compute subproblems in **increasing numeric order**, then every subproblem will be computed **before we need it**

Dynamic Programming Implementation: **nim/nim.py**

```
def solveall():
  # for each losing state, find winning states that reach it
  for j in range(len(self.wins) - 1): # nothing reaches starting state
    if not self.wins[j]: # loss, so find all psns that reach j
      cj = self.crd(j) # convert index to list of pile sizes
      for x in range(len(cj)):
        cjcopy = deepcopy(cj)
        for t in range(1 + cj[x], 1 + self.dim[x]): # `dim` is starting pile

          cjcopy[x] = t
          pjc = self.psn(cjcopy) # convert pile sizes to index
          self.wins[pjc], self.winmove[pjc] = True, j
```

Position: (2,3,4)
Start position: (3,5,7)



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  - xorsum(3,5,3) = 101b = 5
  - $\begin{matrix} 3 = 011 \\ 5 = 101 \\ 3 = 011 \end{matrix}$ and $\begin{matrix} 0 \oplus 1 \oplus 0 = 1 \\ 1 \oplus 0 \oplus 1 = 0 \\ 1 \oplus 1 \oplus 1 = 1 \end{matrix}$

Nim Formula Theorem

Theorem: A Nim position with pile sizes $p_1, \ldots, p_k$ is **losing** iff

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3. $\mathrm{xorsum}(0,0,\ldots,0) = 0$
4. Any position that has $(0,0,\ldots,0)$ as a child is a **winning** position

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1	0	0	1	0
1	1	0	1	0
1	0	1	1	1
0	1	1	0	0
1	1	0	1	1
0	0	0	1	1
0	1	0	1	1
4	3	2	1	0

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(a) Let d be the digit number of the most significant 1 in the binary representation of $\mathrm{xorsum}(P)$

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$d = 3$

Nim Formula Theorem Proof Step 2

2. If $\text{xorsum}(P) \neq 0$, then $\text{xorsum}(C) = 0$ for **some** child C of P

(a) Let d be the digit number of the most significant 1 in the binary representation of $\text{xorsum}(P)$

(b) There exists at least one j such that p_j has a 1 in digit d (**why?**)

(c) $\text{xorsum}(p_1, \ldots, p_{j-1}, p_{j+1}, \ldots, p_k)$ has a 0 in the d-th column (**why?**)

- and *also* in every more-significant column that p_j has a 0 in (**why?**)

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$$\begin{aligned}\text{xorsum}(p_1, \ldots, p_{j-1}, \mathbf{p_j'}, p_{j+1}, \ldots, p_k) &= \text{xorsum}(p_1, \ldots, p_{j-1}, p_{j+1}, \ldots, p_k) \oplus p_j' \\ &= p_j' \oplus p_j' \\ &= 0\end{aligned}$$

	1	0	0	1	0
j	1	1	0	1	0
	1	0	1	1	1
	0	1	1	0	0
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0	**0**	**0**	**0**	**0**

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Nim Formula Theorem Proof Step 5

1. If $\text{xorsum}(\text{P}) = 0$, then $\text{xorsum}(C) \neq 0$ for **every** child C of P
2. If $\text{xorsum}(P) \neq 0$, then $\text{xorsum}(C) = 0$ for **some** child C of P
3. $\text{xorsum}(0,0,\ldots,0) = 0$
4. Any position that has $(0,0,\ldots,0)$ as a child is a **winning** position

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Inductive hypothesis: suppose that for all positions P within k steps of $(0,\ldots,0)$,
$(\text{xorsum}(P) \neq 0 \implies P$ is winning) and $(\text{xorsum}(P) = 0 \implies P$ is losing)

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Then for all P that are within $k+1$ steps of $(0,\ldots,0)$:

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Then for all P that are within $k + 1$ steps of $(0,\ldots,0)$:

(a) $\mathrm{xorsum}(P) \neq 0$:

- by (2), $\mathrm{xorsum}(C) = 0$ for **some** child C
- C is within k of $(0,\ldots,0)$, so by IH C is **losing**
- So P has **at least one losing child**, and is therefore **winning**

Nim Formula Theorem Proof Step 5

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(b) $\text{xorsum}(P) = 0$:

- by (1), $\text{xorsum}(C) \neq 0$ for **all** children C
- every C is within k of $(0,\ldots,0)$, so by IH every C is **winning**
- So **all of P's children are winning**, hence P is **losing**

Nim Formula Theorem Proof Step 5

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Then for all P that are within $k+1$ steps of $(0,\ldots,0)$:

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- every C is within k of $(0,\ldots,0)$, so by IH every C is **winning**
- So **all of P's children are winning**, hence P is **losing**

Base case:

(a) By (3), every position P within 0 of $(0,\ldots,0)$ has $\mathrm{xorsum}(C) = 0$, and by definition it is a losing position

(b) By (1), every position P within 1 of $(0,\ldots,0)$ has $\mathrm{xorsum}(C) \neq 0$, and by (4) they are all winning $\blacksquare$

Using the Nim Formula

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- **Checking** if a position P is **winning** or **losing**: Simply compare $\text{xorsum}(P)$ to 0

Using the Nim Formula

- **Checking** if a position P is **winning** or **losing**: Simply compare $\mathrm{xorsum}(P)$ to 0
- Finding a **winning move**:
  1. Compute $\mathrm{xorsum}(P)$
  2. Find d and a pile j with a 1 in digit d
     - Equivalently: a pile j with $p_j \geq \mathrm{xorsum}(p_1, \ldots, p_{j-1}, p_{j+1}, \ldots, p_k)$
  3. Set $p'_j = \mathrm{xorsum}(p_1, \ldots, p_{j-1}, p_{j+1}, \ldots, p_k)$
  4. Take $p_j - p'_j$ stones from pile j
     - i.e., move to position $(p_1, \ldots, p_{j-1}, \mathbf{p'_j}, p_{j+1}, \ldots, p_k)$

Nim Formula Implementation: **nim/nim.py**

```python
def xorsum(L):
  xsum = 0
  for j in L:
      xsum ^= j
  return xsum
```

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(x ^ y) is the Python syntax for $(x \oplus y)$

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```python
def nimreport(P): # report all winning nim moves from P, use formula
  total = xorsum(P)
  if total==0:
    print(' loss')
    return
  for j in P:
    tj = total^j
    if j >= tj:
      print(' win: take',j - tj,'from pile with',j)
```

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```

Note: $\text{xorsum}(p_1, \ldots, p_{j-1}, p_{j+1}, \ldots, p_k) = \text{xorsum}(p_1, \ldots, p_k) \oplus p_j$ **(why?)**

Summary

- Nim can be solved using **dynamic programming**
  - recursion with memoization (i.e., recursion with transposition table)
  - bottom-up
- The **Nim formula** allows us to **directly compute** whether a Nim move is **winning**
  - *Without* having to solve all the subpositions!
- Can directly compute a **winning move** from any winning position
  - Again, without having to solve subpositions
  - Or even consider every possible child