Solving Small Hex Games

CMPUT 355: Games, Puzzles, and Algorithms

Lecture Outline

1. Logistics & Recap
2. Hex Theorem, redux
3. Applying virtual connections

Logistics

- **Practice questions #4** will be released this Friday (**Mar 6**)
    - Solutions will be released on Tuesday (Mar 10)
- **Quiz 4** is **next Friday** (Mar 13)
    - *Coverage:* up to and including **Mar 6** (next lecture)
    - Bring your student ID!
    - No calculators or other devices
    - The quiz will be run by 3 TAs (I will be in the hospital)
- **TA Office hours:** every Thursday 1pm-2pm in **UCOMM-3-136**

The Hex Theorem

Theorem: In the game of Hex on an $N \times N$ board, there exists a strategy that guarantees a win for Player 1 (Black).

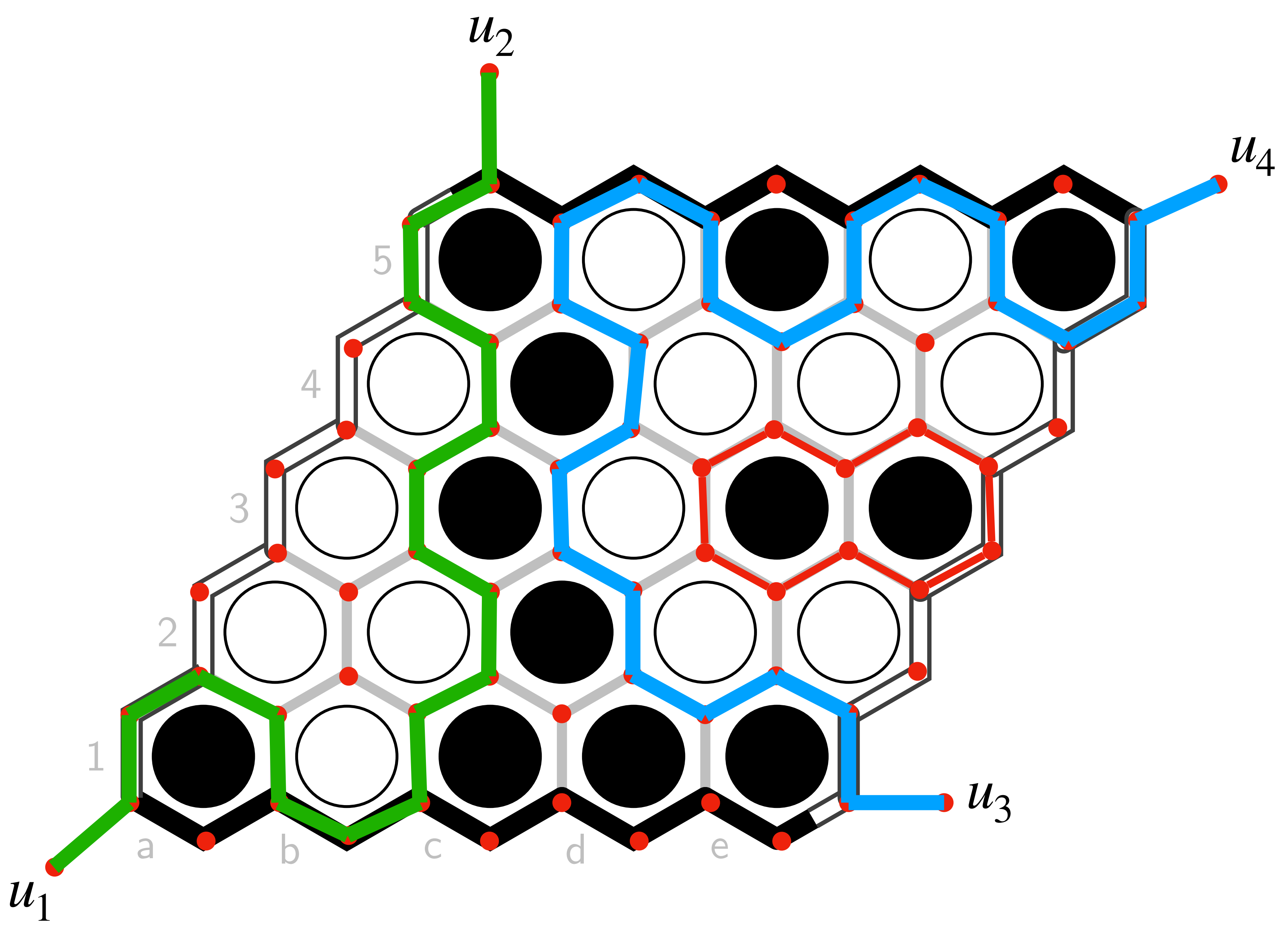
Proof sketch:

1. An extra i-stone never makes player i worse off.
2. No game of Hex is a draw.
3. In $n \times n$ Hex, Player 2 does not have a winning strategy by a "strategy stealing" argument.

Hex Theorem Proof: Step 2b

2. No game of Hex is a draw.

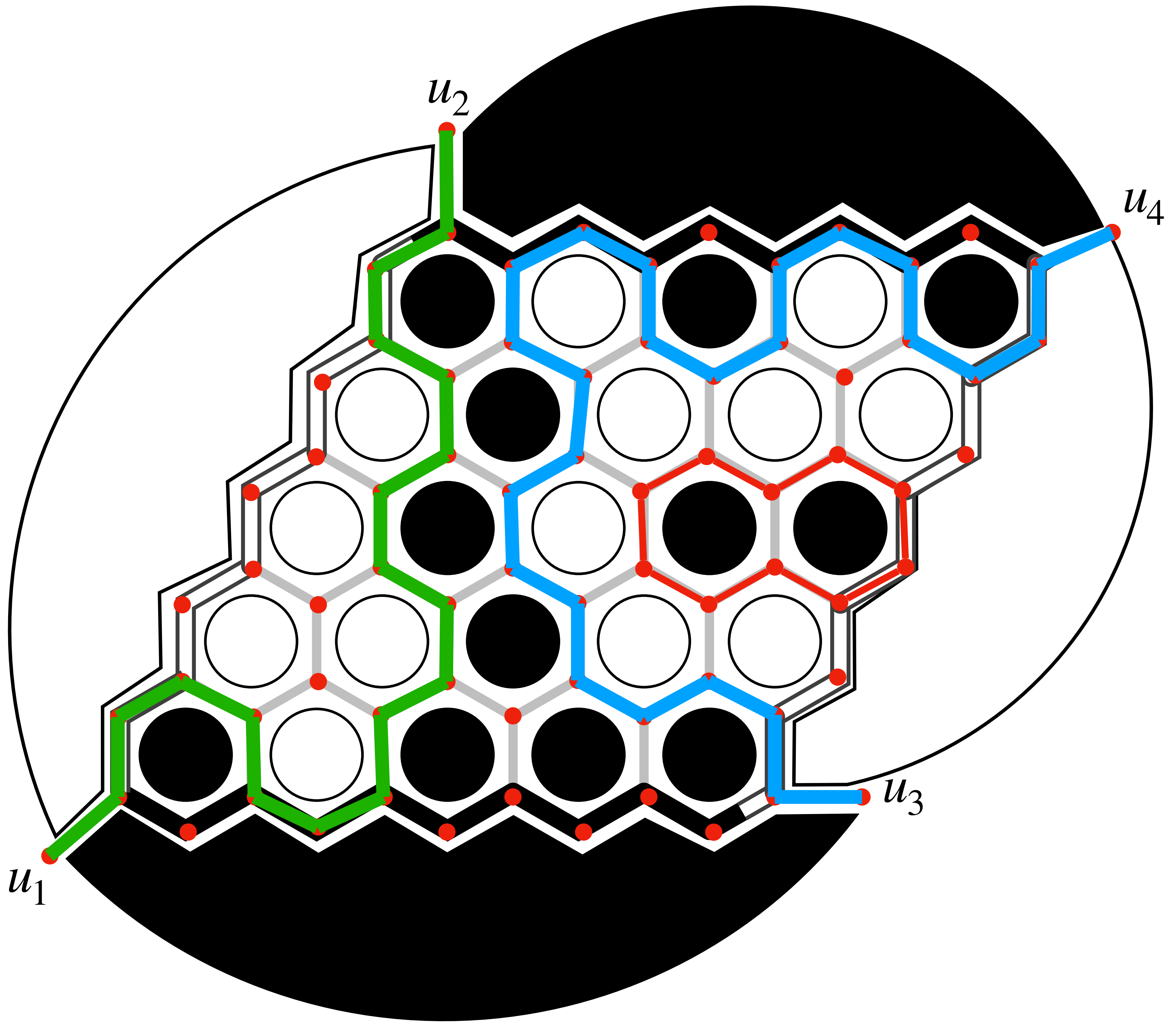
- **Lemma A:** Every graph $G = (V, E)$ of degree 2 or less is a union of disjoint graphs, each of which is either a **simple path**, a **simple cycle**, or an **isolated vertex**.
- From Hex position, construct a graph $G = (V, E)$:
  - Vertices are corners of hex cells (not the cells themselves)
    - Plus four extra nodes u_1, u_2, u_3, u_4 representing the corners
  - Edges are the sides of the hex cells
    - Plus four extra edges e_1, e_2, e_3, e_4 linking u_j to the sides
- Now take a subset E' of the edges to construct $G' = (V, E')$:
  - Include all edges between a Black stone and a White stone
  - Include all Black edges adjacent to White stone and vice versa
  - Include e_1, e_2, e_3, e_4 as well
- G' has degree 2:
  - Nodes surrounded by **same colour** have no edges (degree 0)
  - Nodes surrounded by **two of one colour, one of another** have 2 incoming edges (degree 2)
  - Each of u_1, u_2, u_3, u_4 have degree 1
- So G' contains two simple paths, linking pairs of u_j's (**why?**)



Hex Theorem Proof: Step 3b

2. No game of Hex is a draw.

- **Lemma A:** Every graph $G = (V, E)$ of degree 2 or less is a union of disjoint graphs, each of which is either a **simple path**, a **simple cycle**, or an **isolated vertex**.
- G' contains two simple paths, linking pairs of u_j's
- Claim: $u_1 \rightsquigarrow u_2$ means that Black has a winning path
  - (i) Think of the borders as giant single Hex tiles
    - So two Black stones that touch $\mathbf{top}$ are part of the same **group** (in the usual Hex sense, not in G')
  - (ii) Now consider the set of stones that are **adjacent** to the path from u_1 to u_2
    - This set of stones are all part of the **same group** (**why?**)
  - (iii) At least one Black stone touches the $\mathbf{bottom}$ border (**why?**)
    - Similarly, at least one Black stone touches the $\mathbf{top}$ border
  - (iv) The stones that touch the $\mathbf{bottom}$ border are all part of the path-adjacent group
    - because the path is adjacent to the bottom "big" Black stone, and everything that touches the $\mathbf{bottom}$ border is adjacent to it also
    - By the same argument, the path-adjacent group is adjacent to the $\mathbf{top}$ border
  - (v) So there is a group of stones that are adjacent to both $\mathbf{top}$ and $\mathbf{bottom}$
- $u_1 \rightsquigarrow u_3$ means that White has a winning path by an analogous argument
- $u_1 \rightsquigarrow u_4$ is not possible
  - It would have to cross the path $u_2 \rightsquigarrow u_3$ at some point
  - but then it would not be a **simple path** (because some interior nodes would have degree 4)
- SO: Either $u_1 \rightsquigarrow u_2$ and Black wins, or $u_1 \rightsquigarrow u_3$ and White wins. ■

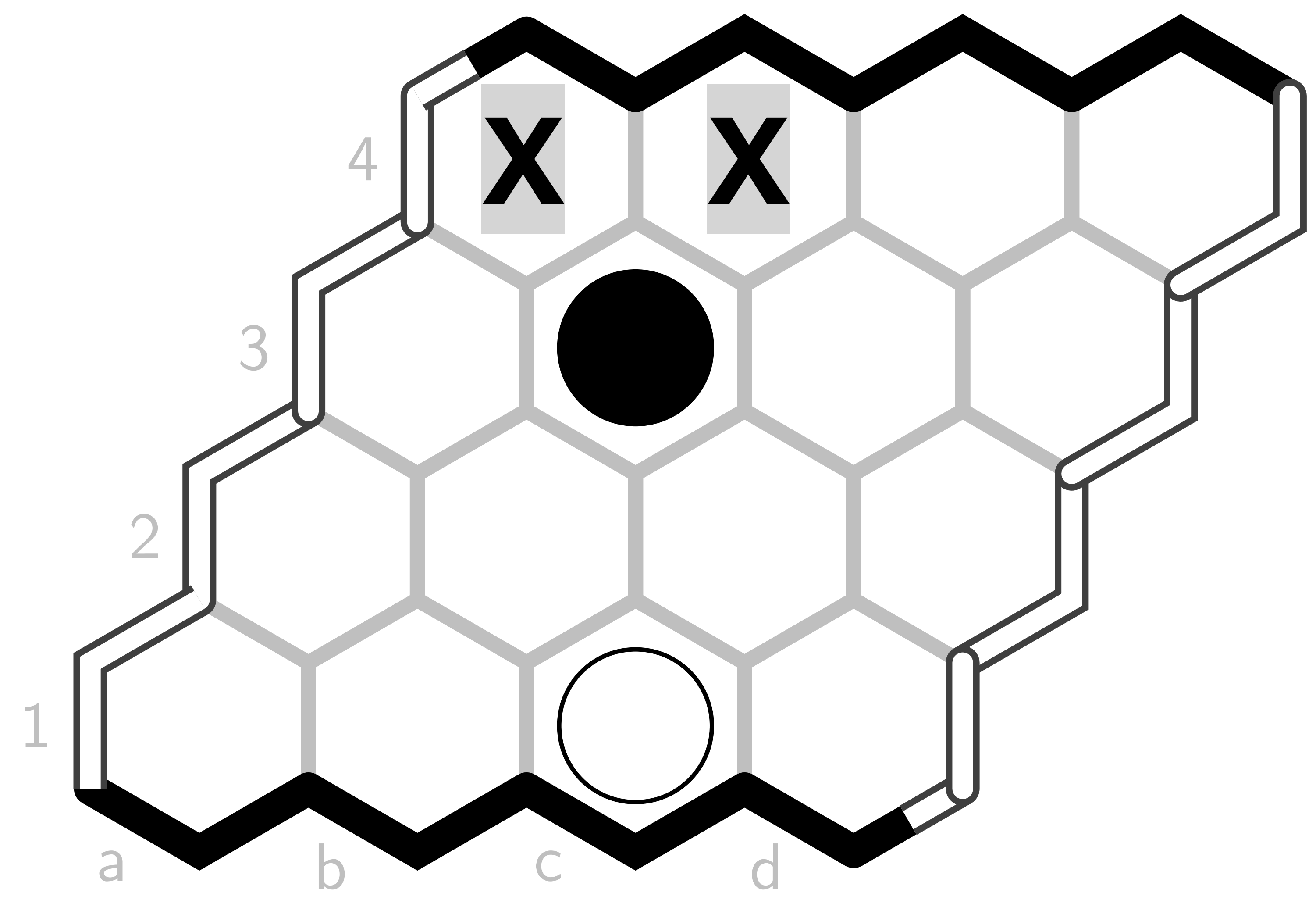


Hex Theorem Proof: Step 3

1. An extra i-stone never makes player i worse off.
2. No game of Hex is a draw.
3. In $n \times n$ Hex, Player 2 does not have a winning strategy by a "strategy stealing" argument.
   (i) **Suppose not!** So Player 2 has a winning strategy it can play.
   (ii) Player 1 plays an arbitrary stone and "forgets" about it
   (iii) Player 2 starts their strategy
   (iv) Player 1 follows Player 2's winning strategy!
      - By (1), the initial stone cannot make Player 1 worse off
      - So it can follow the winning strategy as if it were Player 2
   (v) But this is a contradiction, so Player 2 does *not* have a winning strategy
      - i.e., Player 2 cannot force a win
      - so Player 1 can force either a **win** or a **draw**
   (vi) By (2), nobody can force a draw
   (vii) So Player 1 can force a win. ■

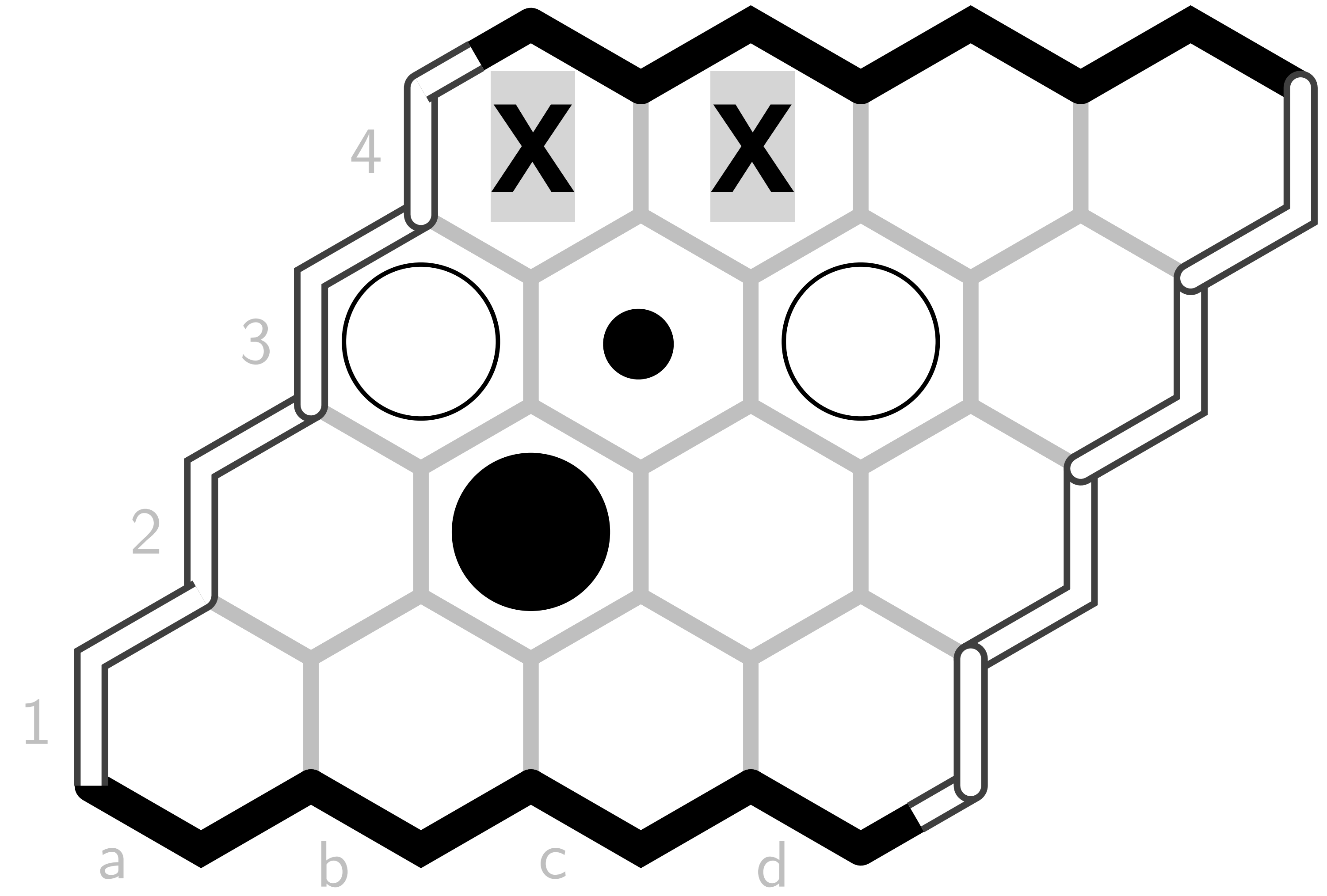
Recap: Full Connection

- **Virtual connection:**
  A player **can** connect two cells, but they haven't **yet**
- **Full connection:** A player can connect two cells whether they are the **current** player to move *or* the **next** player to move.
- **Example:** Black can force a connection from **b3** to the top edge **even if it's White's turn next** (**how?**)



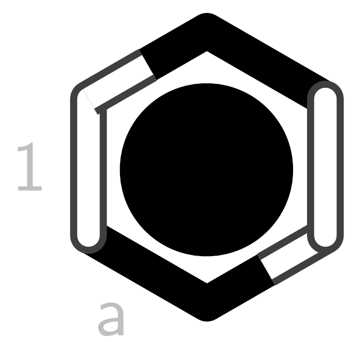
Recap: Semi-Connection

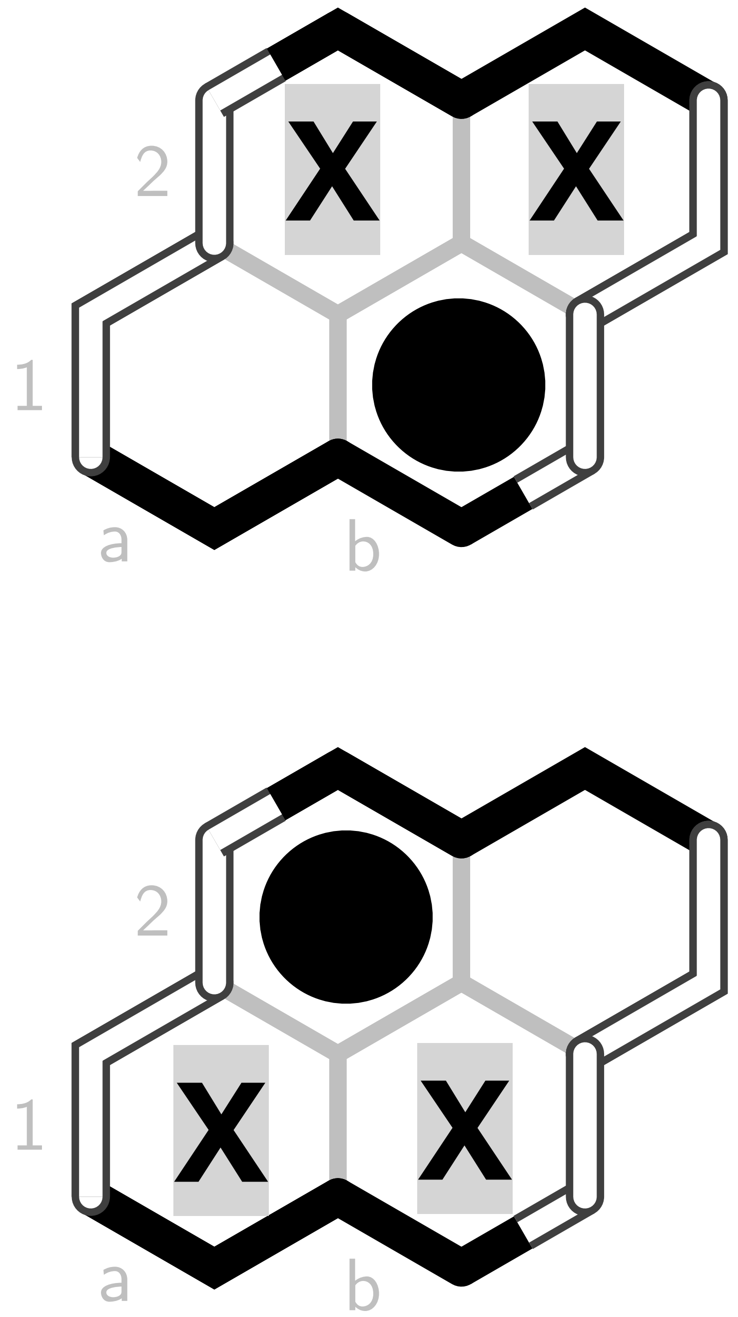
- **Virtual connection:**
  A player **can** connect two cells, but they haven't **yet**
- **Semi connection:** A player can force a connection between two cells, but only if they are the **current** player to move
- **Example:** Black can force a connection between b2 and the top edge, but only if it is Black's turn.

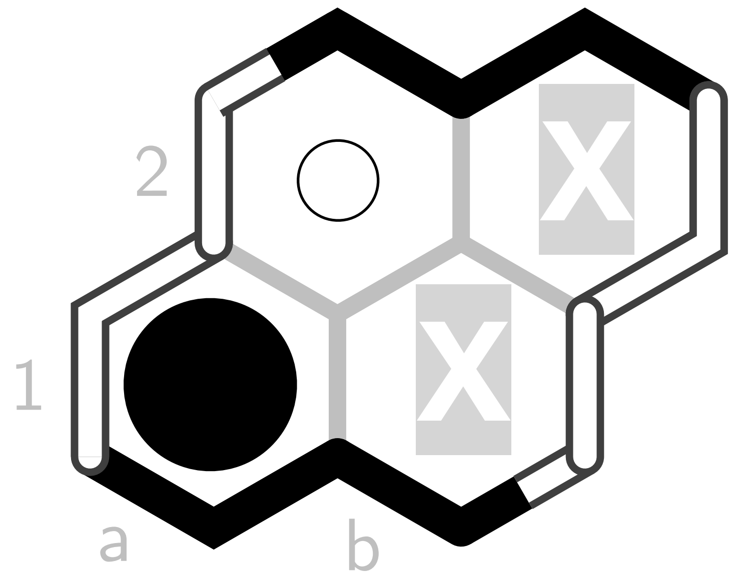


Warm-up Puzzles

Can you find a winning opening move for Black?

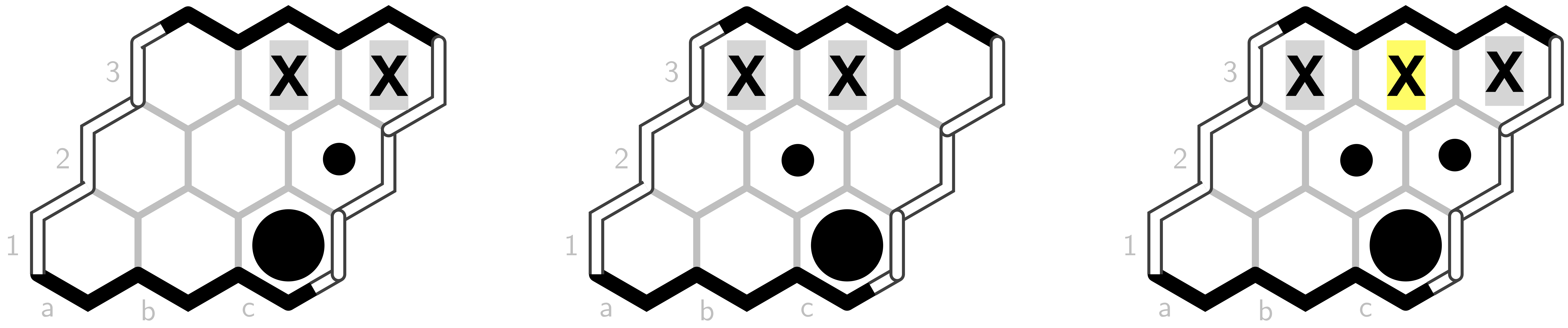






- Even in the 2x2 case, **not every opening** is a win for Black!
  - Opening in an **obtuse** corner is a win
  - But opening in an **acute** corner gives White a win threat!

3x3 Win Threats



Threat A

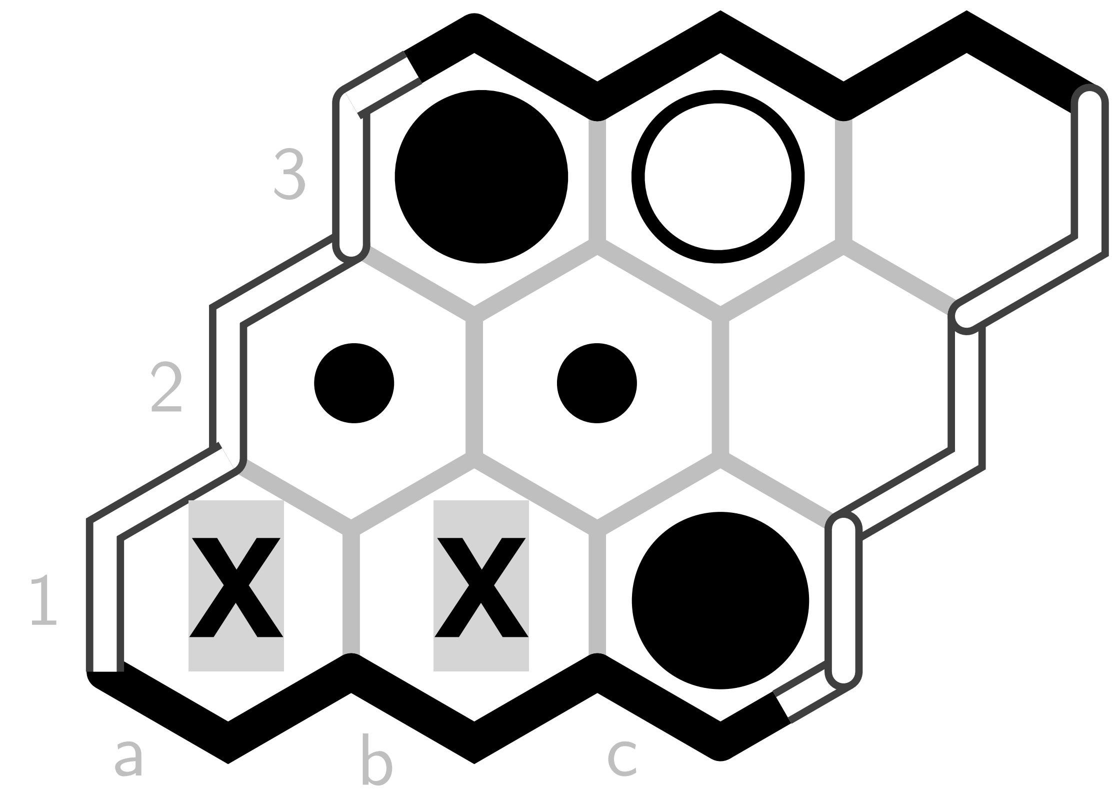
Threat B

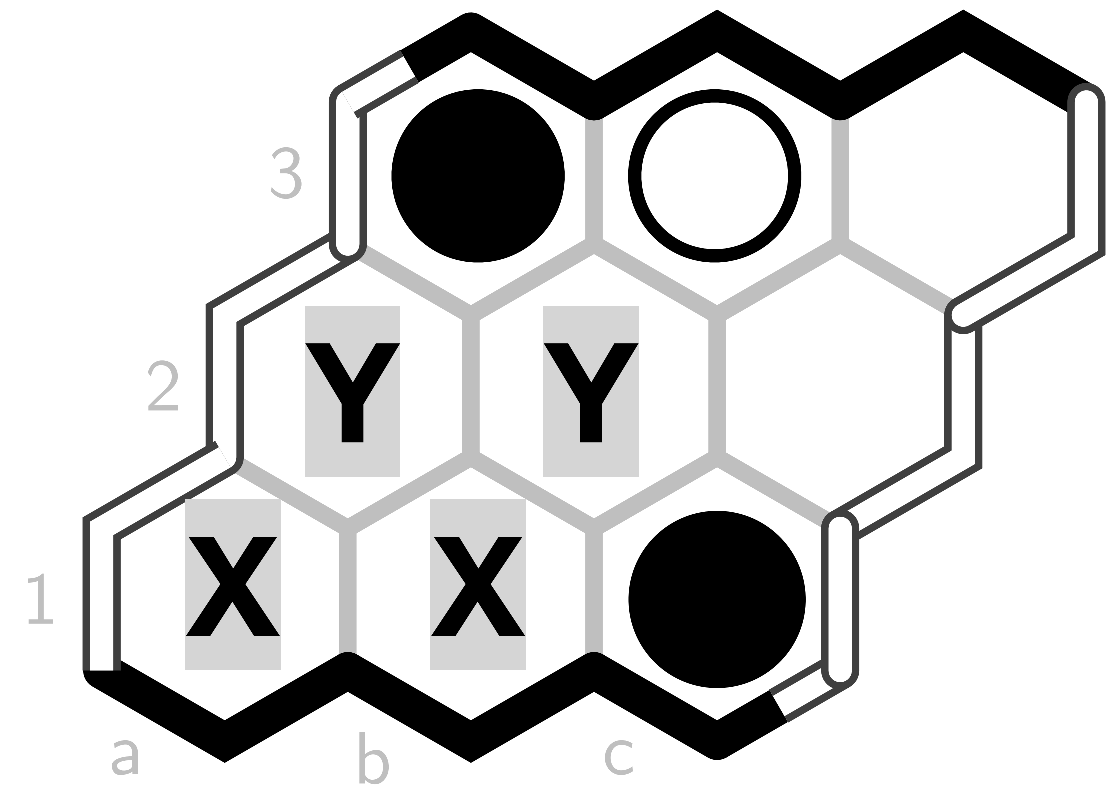
Non-empty intersection

- **Question:** Where should White play?
- Does Black have a **win-threat**?
    - White can block **Threat A** by playing on one of {**c2**, **b3**, **c3**}
    - White can block **Threat B** by playing on one of {**b2**, **a3**, **b3**}
- **Question:** Can White block **both threats** at once?

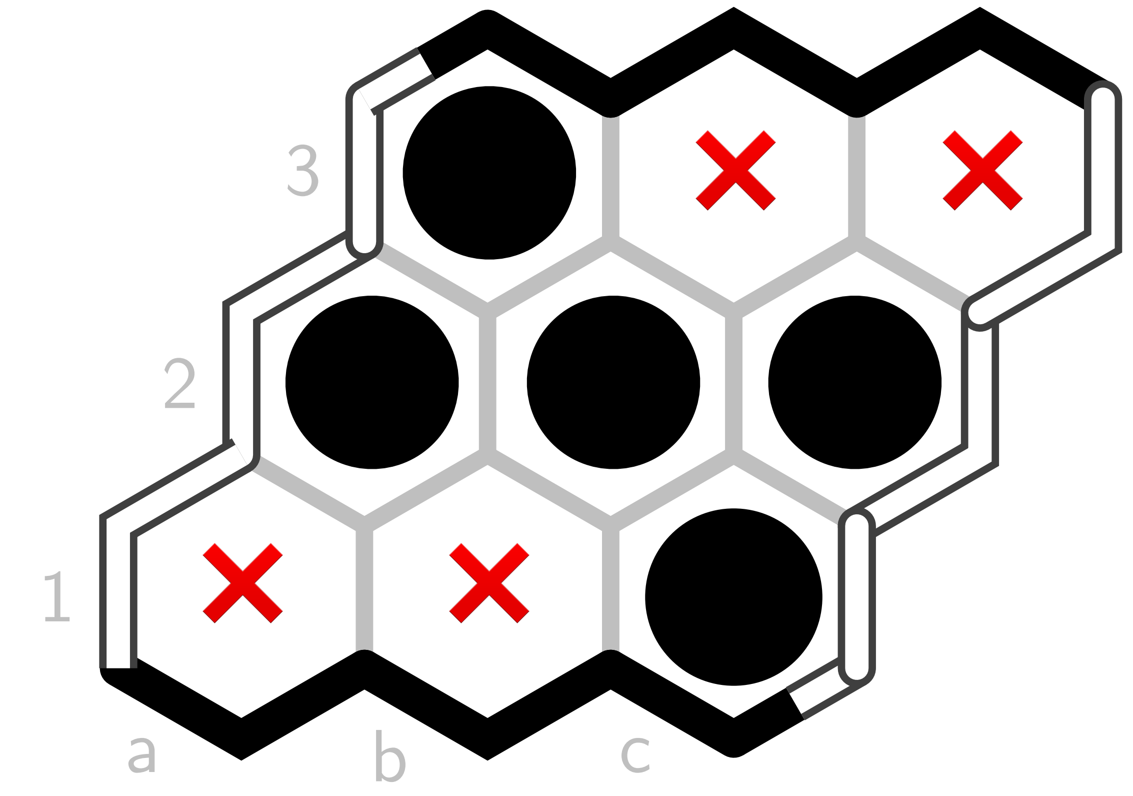
Pairing Strategies

- Playing **b3** blocks Black's immediate win threats
- Black still has a winning strategy:
  - **a3** has virtual semi-connection to **c1**
  - and **a2** has virtual full connection from **bottom** to **a2** (which has virtual semi-connection to **a3**)
- So: If White plays one of {**a2**, **b2**}, play **the other one**
- If White plays one of {**a1**, **b1**}, play the other one
- This is called a **pairing strategy**:
  - Group empty cells into **pairs**
  - Whenever opponent plays one cell of a pair, reply on the **other cell**

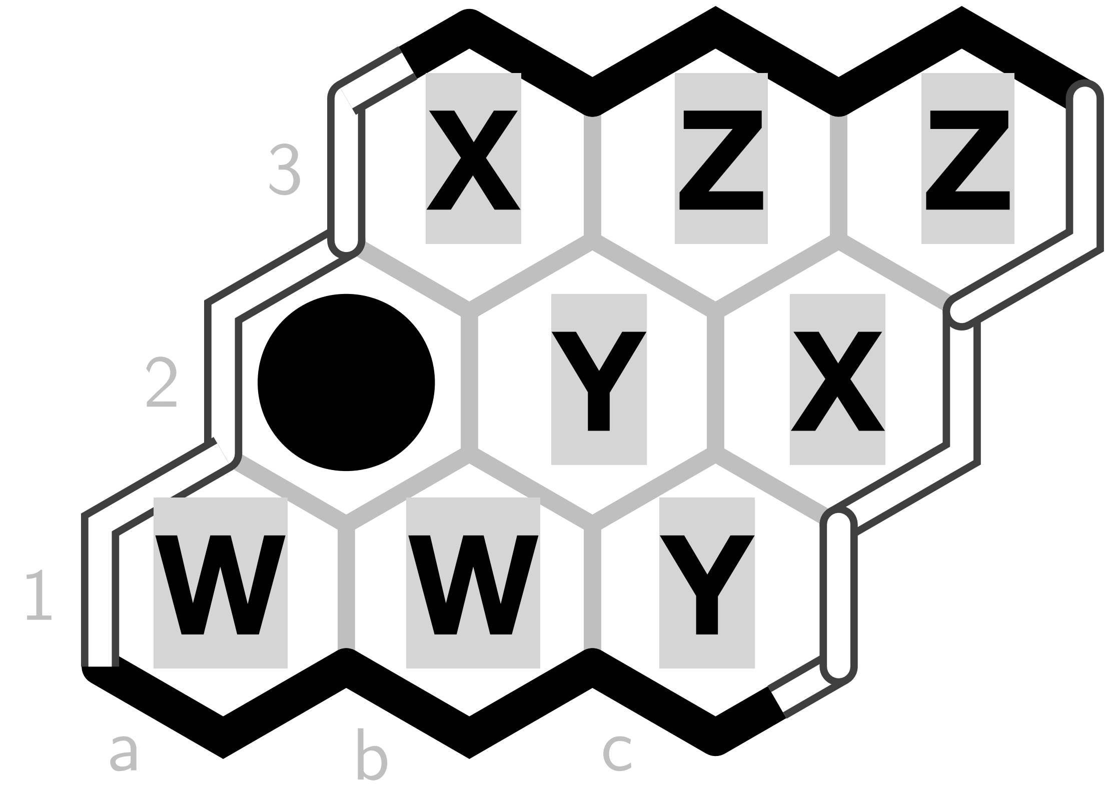


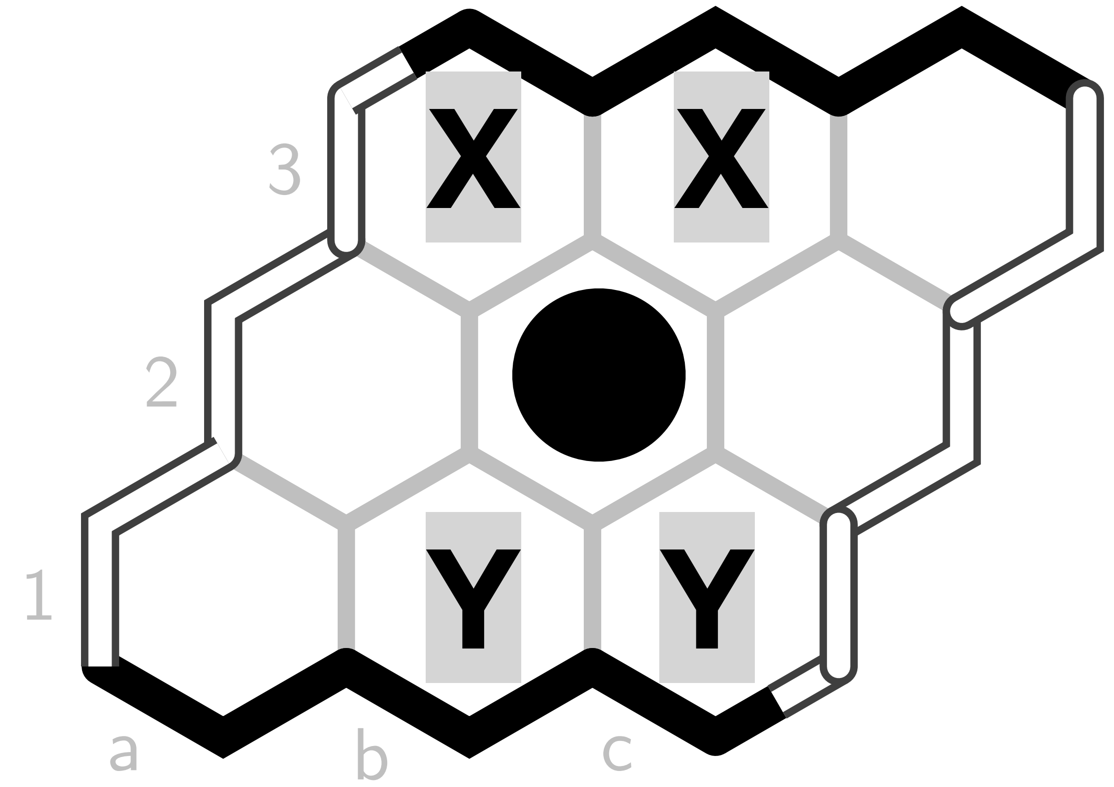


Solving 3x3 Hex



- We just saw that c1 is a winning opening
  - **Question:** What else must be, by isomorphism?
- Is this a winning opening?
  - White has a **win threat**: **a3** has virtual full connection to **c2**
  - Virtual full connection from **a2** to **bottom**
  - Virtual full connection from **c2** to **top**
  - Virtual semi-connections:
    - **c2** to **bottom**
    - **c2** to **a2**
  - Yes, this is a winning opening (so is **c2** by isomorphism)
- **b2** is a winning opening (**why?**)
- Remaining cells are **not winning** (**why?**)





Summary

- **Hex Theorem:**
  - $n \times n$ Hex is a **win for Black** for all n
  - **Strategy-stealing argument** that White cannot force a win
  - *Crucial step:* Hex cannot be a draw
- **Virtual connections** allow us to construct winning strategies
  - Often without searching the entire state graph
  - Strategies often take the form of **pairing strategies**
  - This is sufficient to solve 3×3 Hex completely