Hex Analysis

CMPUT 355: Games, Puzzles, and Algorithms

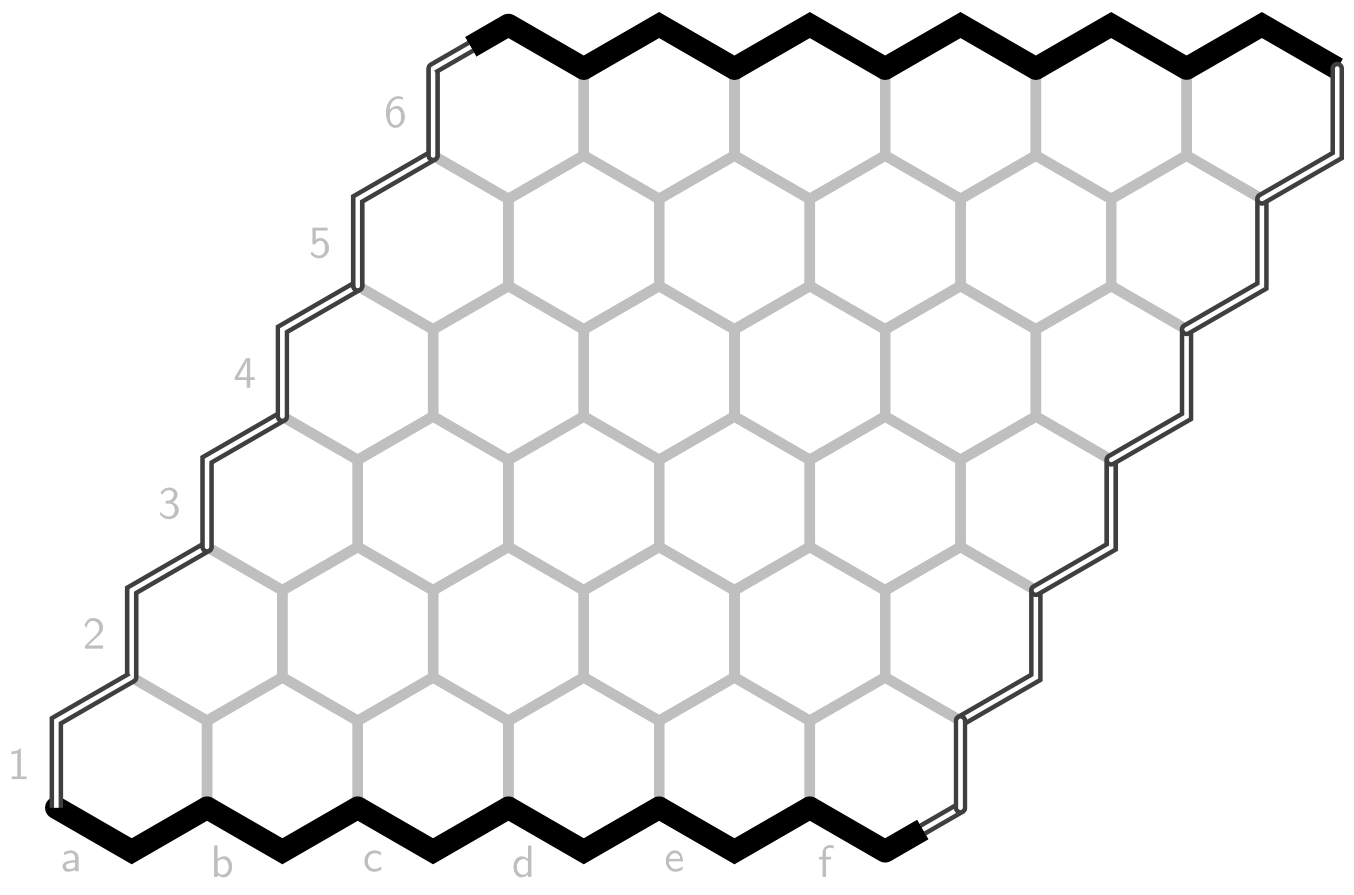
Lecture Outline

1. Logistics & Recap
2. Hex Theorem
3. Virtual Connections

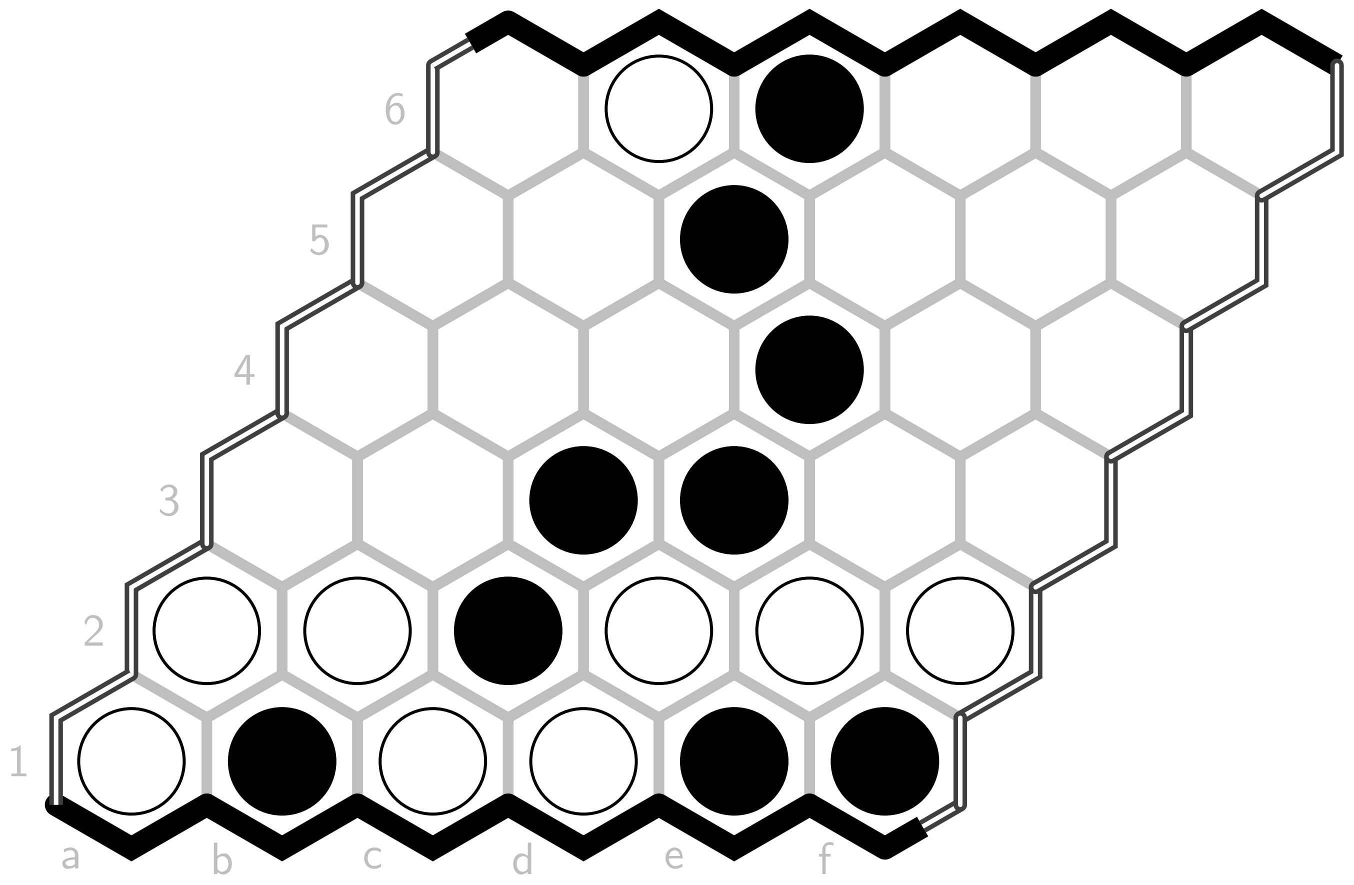
Recap: Hex Rules

- There are two players, Black and White
- Each player makes a **move** in alternation, starting with Black
  - A move is placing a stone in an Empty hexagonal cell
- Two cells are **adjacent** if they share a side
  - Each cell has 2-6 neighbours
- Two facing **borders** of the board are Black, the other two edges are White
  - Each bottom cell is adjacent to the bottom border, etc.
  - Each **corner** cell is thus adjacent to **two** borders
- The game ends when one player has joined the two edges of their own colour with a path of stones
  - The player who joins their edges **wins**

Empty Hex board



Winning position for White



Recap: Go is Hard

- **2x2 Go:**
  - Can be solved using **just alpha-beta search**
    - Without alpha-beta: hopeless
  - **Pass-first** move ordering dramatically reduces the search space
- **3x3 Go** is **significantly harder**
  - Approximately 10^{1100} possible games
  - So 2x2 Go is right at the edge of feasibility for alpha-beta-based techniques
- **Question:** How is it possible for there to be so many games when there are only 9 possible moves?

Hex is Easier But Nontrivial

- Hex is much **easier** to analyze than Go
    - No captures means no cycling in the state graph
    - Several approaches allow for reducing the search space
        - Today: virtual connections
    - $N \times N$ Hex is known to be a first-player win
        - Today: proof
- But, Hex is still **nontrivial**
    - We don't know the winning strategy beyond 10×10 Hex
    - Unlike Nim, where analytical results give us an immediate winning strategy

The Hex Theorem

Theorem: In the game of Hex on an $N \times N$ board, there exists a strategy that guarantees a win for Player 1 (Black).

Proof sketch:

1. An extra i-stone never makes player i worse off.
2. No game of Hex is a draw.
3. In $n \times n$ Hex, Player 2 does not have a winning strategy by a "strategy stealing" argument.

Hex Theorem Proof: Step 1

1. An extra i-stone never makes player i worse off.

- Suppose i is playing a strategy s_i that maps from position to next move
- Place an i-stone on an arbitrary empty cell (call it cell x)
    - Extra i stone will never remove winning options for i (no winning path gets blocked)
    - Extra i stone will never add winning options for $-i$ (no winning path gets created)
- Whenever called upon to play on position P, let P_{-x} be P with stone removed from x
    - if $s_i(P_{-x}) \neq x$, then play $s_i(P_{-x})$; nothing is different except there's an extra Black stone
    - else if $s_i(P_{-x}) = x$, then:
        - Play on some empty cell y
        - Assign $x := y$

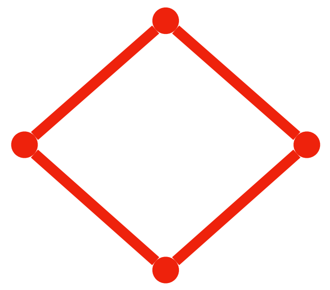
Hex Theorem Proof: Step 2a

2. No game of Hex is a draw.

- It is sufficient to show that any position where every cell contains a stone is a win for one of the players (**why?**)
- **Lemma A:** Every graph $G = (V, E)$ of degree 2 or less is a union of disjoint graphs, each of which is either a **simple path**, a **simple cycle**, or an **isolated vertex**.
- Proof by induction on $|E|$
  - *Base case:* Graph with 0 edges is union of $|V|$ isolated nodes
  - *Inductive hypothesis (IH):* Lemma holds when $|E| = k$
  - *Inductive step:* In a graph with $k + 1$ edges, remove arbitrary edge $(u, v) \in E$
    - Resulting graph is union of simple paths, simple cycles, isolated nodes by IH
    - In this graph, u has degree at most 1, v has degree at most 1 (so not in a cycle)
    - Adding (u, v) back joins two isolated vertices, or two simple paths at their endpoints, or a simple path at endpoint and an isolated vertex, or the two ends of a simple path (making a simple cycle)
      - and doesn't effect any of the subgraphs that don't include $\{u, v\}$

Isolated vertex

Simple path

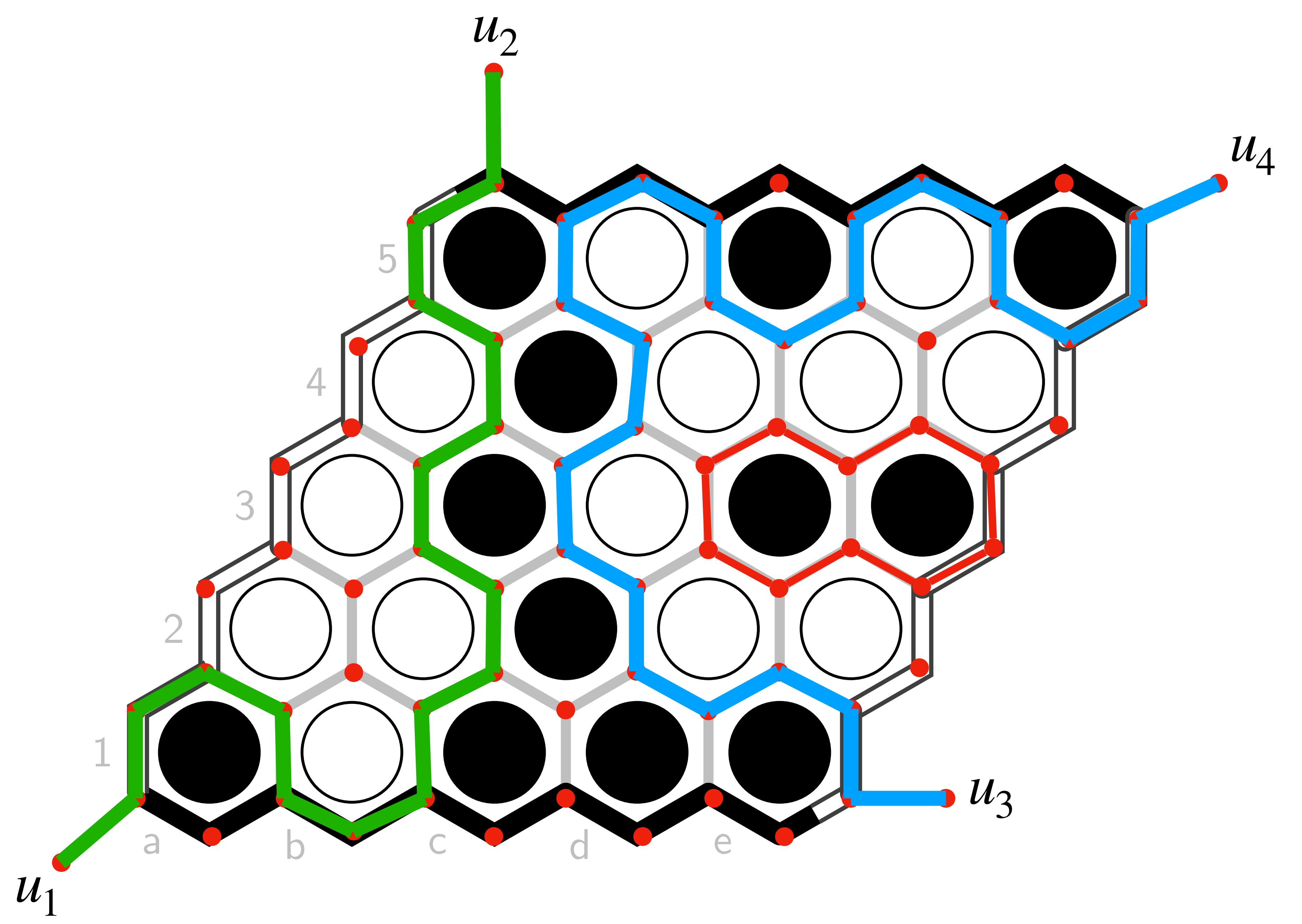


Simple cycle

Hex Theorem Proof: Step 2b

2. No game of Hex is a draw.

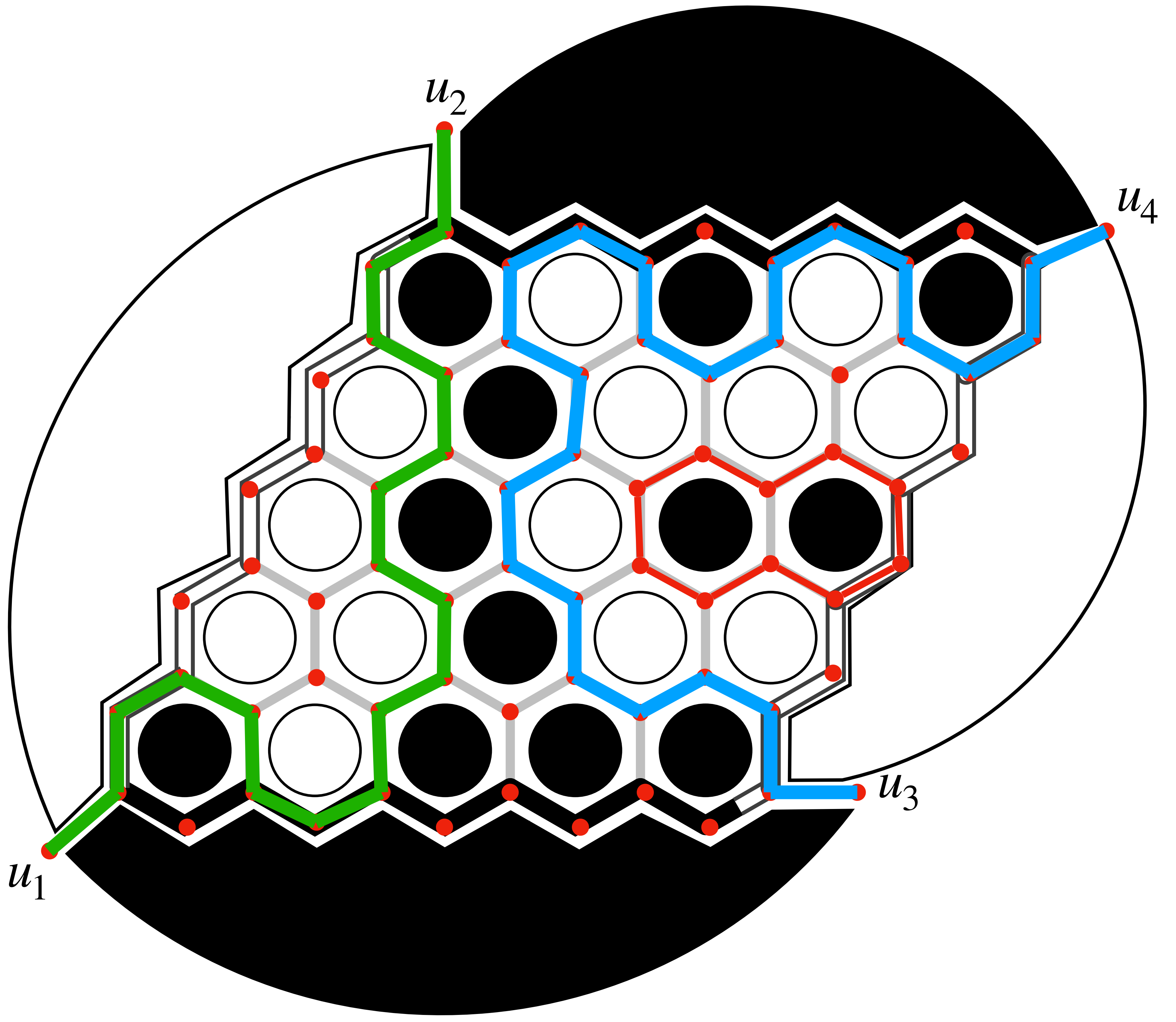
- **Lemma A:** Every graph $G = (V, E)$ of degree 2 or less is a union of disjoint graphs, each of which is either a **simple path**, a **simple cycle**, or an **isolated vertex**.
- From Hex position, construct a graph $G = (V, E)$:
  - Vertices are corners of hex cells (not the cells themselves)
    - Plus four extra nodes u_1, u_2, u_3, u_4 representing the corners
  - Edges are the sides of the hex cells
    - Plus four extra edges e_1, e_2, e_3, e_4 linking u_j to the sides
- Now take a subset E' of the edges to construct $G' = (V, E')$:
  - Include all edges between a Black stone and a White stone
  - Include all Black edges adjacent to White stone and vice versa
  - Include e_1, e_2, e_3, e_4 as well
- G' has degree 2:
  - Nodes surrounded by **same colour** have no edges (degree 0)
  - Nodes surrounded by **two of one colour, one of another** have 2 incoming edges (degree 2)
  - Each of u_1, u_2, u_3, u_4 have degree 1
- So G' contains two simple paths, linking pairs of u_j's (**why?**)



Hex Theorem Proof: Step 3b

2. No game of Hex is a draw.

- **Lemma A:** Every graph $G = (V, E)$ of degree 2 or less is a union of disjoint graphs, each of which is either a **simple path**, a **simple cycle**, or an **isolated vertex**.
- G' contains two simple paths, linking pairs of u_j's
- Claim: $u_1 \rightsquigarrow u_2$ means that Black has a winning path
  - (i) Think of the borders as giant single Hex tiles
    - So two Black stones that touch **top** are part of the same **group** (in the usual Hex sense, not in G')
  - (ii) Now consider the set of stones that are **adjacent** to the path from u_1 to u_2
    - This set of stones are all part of the **same group** (**why?**)
  - (iii) At least one Black stone touches the **bottom** border (**why?**)
    - Similarly, at least one Black stone touches the **top** border
  - (iv) The stones that touch the **bottom** border are all part of the path-adjacent group
    - because the path is adjacent to the bottom "big" Black stone, and everything that touches the **bottom** border is adjacent to it also
    - By the same argument, the path-adjacent group is adjacent to the **top** border
  - (v) So there is a group of stones that are adjacent to both **top** and **bottom**
- $u_1 \rightsquigarrow u_3$ means that White has a winning path by an analogous argument
- $u_1 \rightsquigarrow u_4$ is not possible
  - It would have to cross the path $u_2 \rightsquigarrow u_3$ at some point
  - but then it would not be a **simple path** (because some interior nodes would have degree 4)
- SO: Either $u_1 \rightsquigarrow u_2$ and Black wins, or $u_1 \rightsquigarrow u_3$ and White wins. ■



Hex Theorem Proof: Step 3

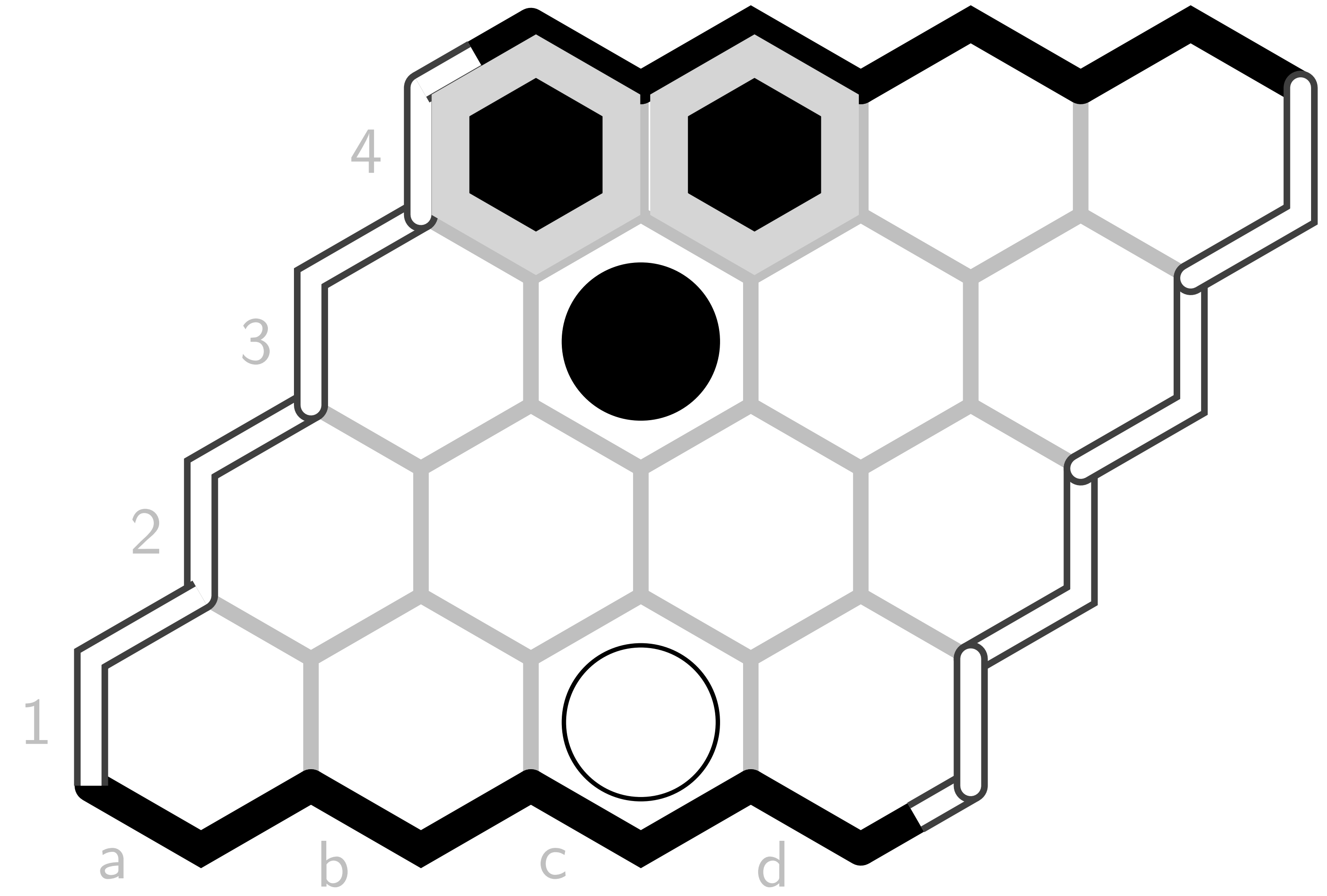
1. An extra i-stone never makes player i worse off.
2. No game of Hex is a draw.
3. In $n \times n$ Hex, Player 2 does not have a winning strategy by a "strategy stealing" argument.
   (i) **Suppose not!** So Player 2 has a winning strategy it can play.
   (ii) Player 1 plays an arbitrary stone and "forgets" about it
   (iii) Player 2 starts their strategy
   (iv) Player 1 follows Player 2's winning strategy!
   - By (1), the initial stone cannot make Player 1 worse off
   - So it can follow the winning strategy as if it were Player 2

   (v) But this is a contradiction, so Player 2 does *not* have a winning strategy
   - i.e., Player 2 cannot force a win
   - so Player 1 can force either a **win** or a **draw**

   (vi) By (2), nobody can force a draw
   (vii) So Player 1 can force a win. ■

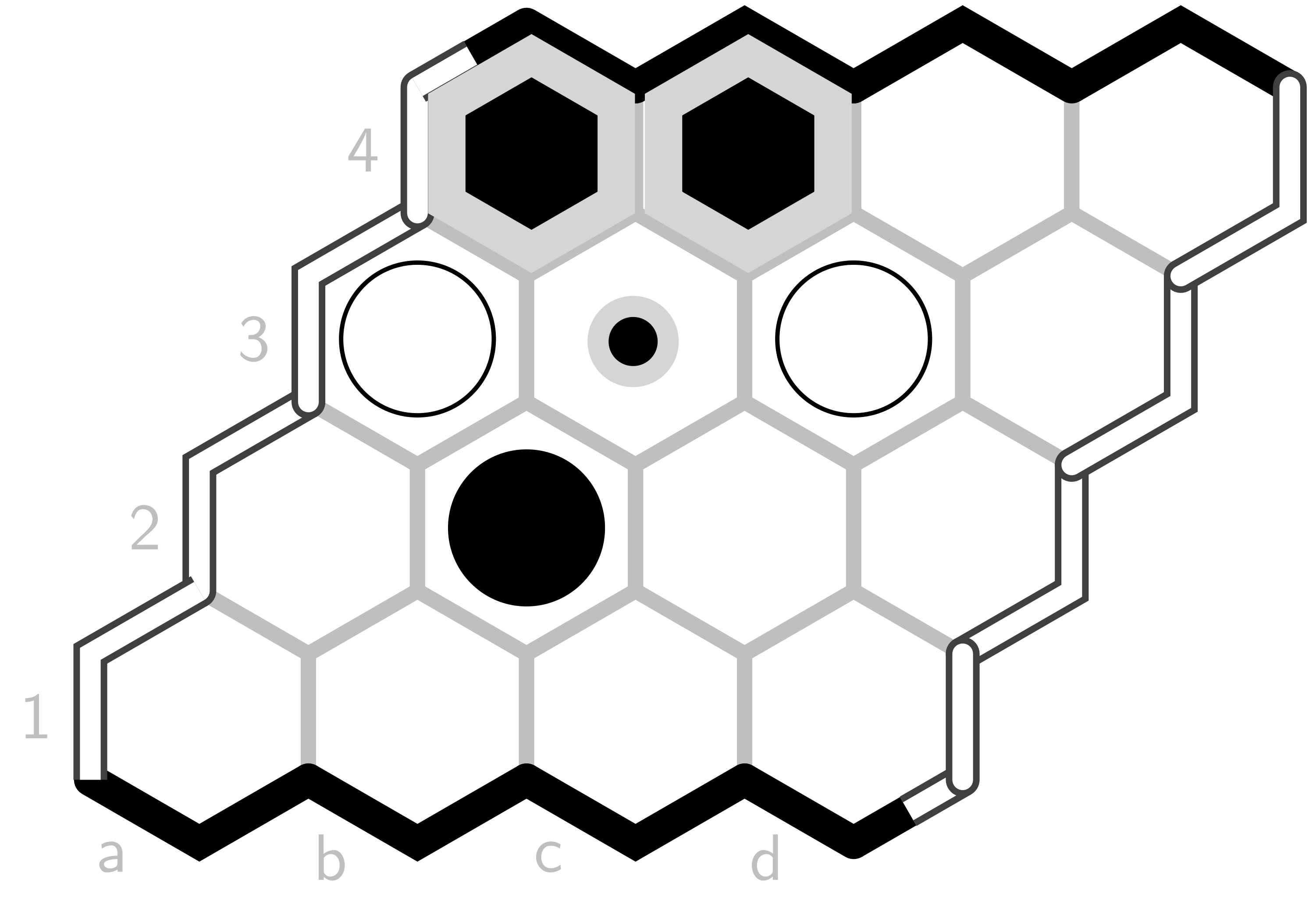
Virtual Connections: Full Connection

- **Virtual connection:**
  A player **can** connect two cells, but they haven't **yet**
- **Full connection:** A player can connect two cells whether they are the current player to move **or** the next player to move.
- **Example:** Black can force a connection from **b3** to the top edge **even if it's White's turn next** (**how?**)



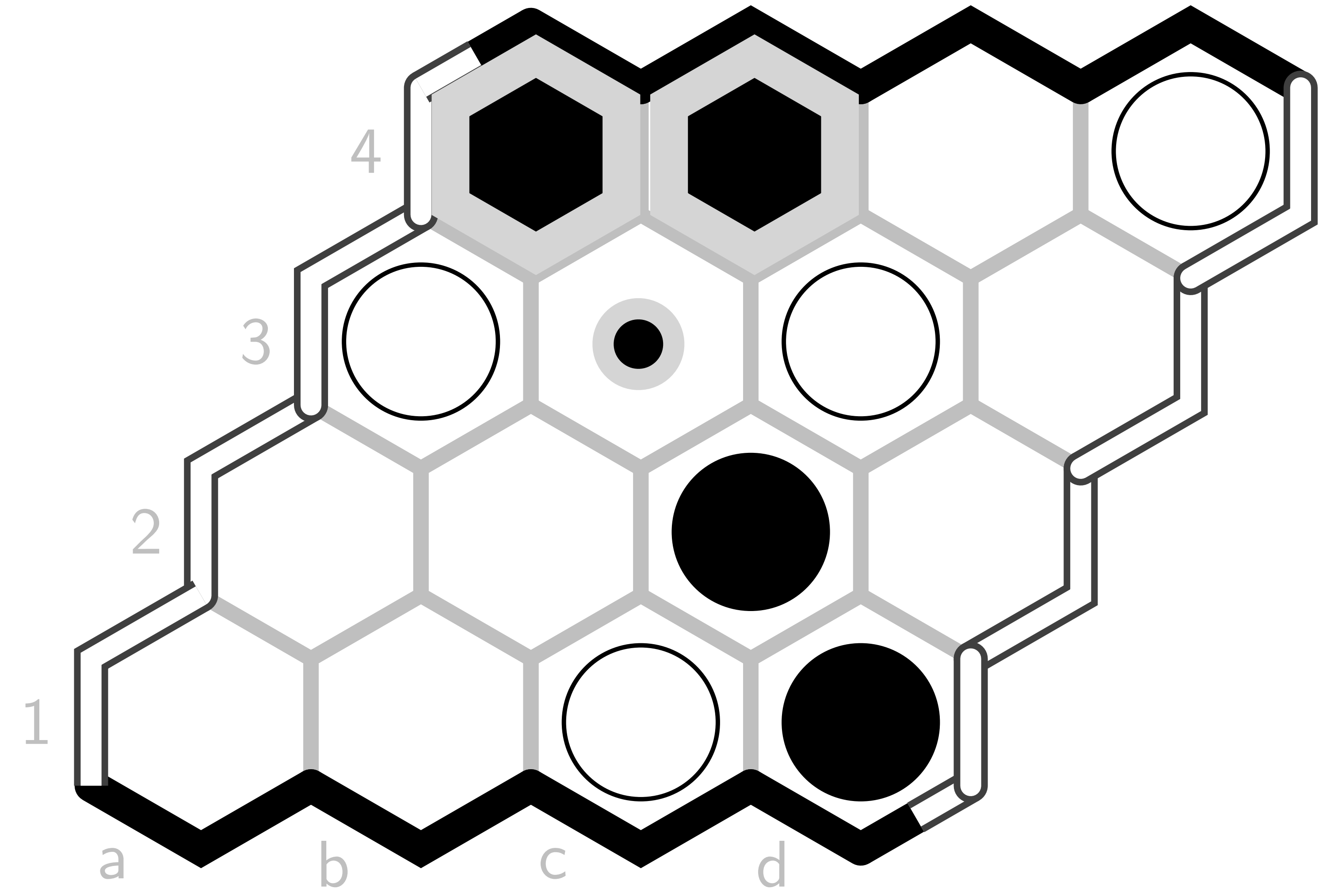
Virtual Connections: Semi-Connection

- **Virtual connection:**
  A player **can** connect two cells, but they haven't **yet**
- **Semi connection:** A player can force a connection between two cells, but only if they are the **current** player to move
- **Example:** Black can force a connection between b2 and the top edge, but only if they move next.



Must-Play Analysis: Preview

- **Question:** How many **possible** next moves does White have in this position?
- **Question:** Where **must** White play in this position?
  - There is a **semi virtual connection** from the **top** edge to **c2** (and from there to the **bottom** edge)
  - If White doesn't prevent this, it has lost the game
  - So it **must** block this connection by playing on one of **b3**, **a4**, **b4**
- Finding virtual connections lets us find **win-threats**
  - This lets us **prune** the search space dramatically



Summary

- Hex is **easier** to analyze than Go, but still **non-trivial**
  - I.e., not as easy to analyze as Nim
- We can show analytically that $n \times n$ **Hex is a win for Black**
  - But the proof is non-constructive; i.e., it doesn't tell us the winning strategy
  - Finding the winning strategy is hard in general
- **Virtual connections** are connections that a player can force
  - **full:** unconditionally
  - **semi:** if they are the current player to move
- Identifying virtual connections enables us to identify **win threats**
  - **Pruning:** If you have a win threat, you don't need to consider other next moves