1. Consider the following normal form game:

| | $B$ | $S$ |
|---|---|---|
| $B$ | $3, 1$ | $0, 0$ |
| $S$ | $0, 0$ | $1, 3$ |

Ballet or Soccer

(a) What are the pure strategy Nash equilibria of this game? Justify your answer.
$(B, B)$ and $(S, S)$; in both cases, if either agent changes actions they will mis-match with the other agent and get a worse utility (0), so neither agent has a utility-improving deviation.

(b) What are the mixed strategy Nash equilibria of this game? Justify your answer.
In a mixed-strategy equilibrium, both agents must be indifferent between any actions they play with positive probability, given the mixed strategy played by the other agent. First, we find the distribution over player 1 (row)'s actions that would make player 2 (column) indifferent between both their actions. We assume that player 1 will play $B$ with probability $p$ and $S$ with probability $1 - p$:

$$\begin{aligned} & pu_2(B,B) + (1-p)u_2(S,B) = pu_2(B,S) + (1-p)u_2(S,S) \\ \iff \quad & p1(1-p)0 = p0 + (1-p)3 \\ \iff \quad & p = 3 - 3p \\ \iff \quad & 4p = 3 \\ \iff \quad & p = \frac{3}{4}. \end{aligned}$$

Similarly, assume that player 2 plays $B$ with probability $q$ and $S$ with probability $1 - q$, and solve for the $q$ that makes player 1 indifferent:

$$\begin{aligned} & qu_1(B,B) + (1-q)u_1(B,S) = qu_1(S,B) + (1-q)u_1(S,S) \\ \iff \quad & q3 + (1-q)0 = q0 + (1-q)1 \\ \iff \quad & 3q = 1 - q \\ \iff \quad & 4q = 1 \\ \iff \quad & q = \frac{1}{4}. \end{aligned}$$

Since each player is indifferent between all (two) of their actions, they cannot have a strictly utility-improving deviation to another strategy; switch from the mixed strategy to either $B$ or $S$ will yield identical utility to that of the mixed strategy.

(c) What is the row player's *expected utility* in each Nash equilibrium of this game?
In $(B, B)$, player 1 gets 3 and player 2 gets 1.
In $(S, S)$, player 1 gets 1 and player 2 gets 3.
In the mixed strategy, player 1 gets

$$u_1(B, q) = qu_1(B, B) + (1 - q)u_1(B, S) = 3q = 3/4,$$

and player 2 gets

$$u_2(p, B) = pu_2(B, B) + (1 - p)u_2(S, B) = 1p = 3/4.$$

(d) Does this game have any *dominant strategies*? If so, which strategy is dominant?
No, it does not have any dominant strategy. For both players, $B$ is a best response to $B$ but not to $S$, and $S$ is a best response to $S$ but not to $B$; neither action dominates the other.

2. Consider the following normal form game:

| | $A$ | $B$ |
|---|---|---|
| $A$ | $3,1$ | $2,0$ |
| $B$ | $0,0$ | $1,3$ |

Artichoke or Banana?

(a) What are the pure strategy equilibria of this game? Justify your answer.
$(A, A)$; neither player has a utility-increasing deviation, since any change would give them both a utility of 0. This is the only pure strategy equilibrium:
From $(A, B)$, player 2 has a deviation to $A$; from $(B, B)$, player 1 has a deviation to $A$; from $(B, A)$, player 1 has a deviation to $A$.

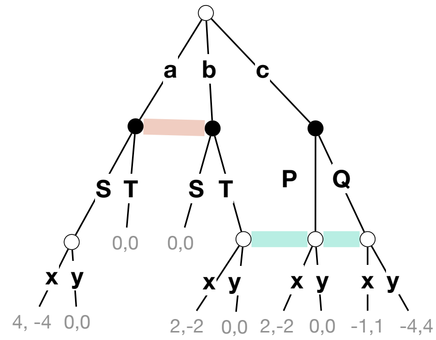
(b) What are the mixed strategy equilibria of this game? Justify your answer.
This game has no mixed strategy Nash equilibrium, because $A$ strictly dominates $B$ for player 1, so there is no way to make player 1 indifferent between their two actions, and given that player 1 plays $A$, player 2's unique best response is also $A$.

(c) Consider the *repeated game* in which Artichoke or Banana is played 5 times in a row. Does this repeated game have an equilibrium in which $B$ is played at least once (by either or both players)? Why or why not?
**No**, by a backward induction argument. In the last game, player 1 has no reason to play anything but $A$; player 2 knows this, and so will only play $A$ as well. But since the outcome of that stage game is known, the same logic applies to game 4, and so on.

(d) Consider the repeated game in which Artichoke or Banana is played infinitely many times, and where the payoffs are combined using a discount rate of $\beta = 0.9$. Does this repeated game have an equilibrium in which $B$ is played at least 25% of the time (by either or both players)? Why or why not?
**Yes.** Consider the Grim Trigger strategy in which the players play $(A, A)$ three times followed by $(B, B)$ once, and repeat this cycle forever. Each player's strategy says to play this cycle unless the other player ever fails to play their part. If either player fails to play their part, then the other player will play a punishment action ($A$ for player 1, $B$ for player 2) until the end of time. This is an equilibrium, since both players get more from the 4-profile cycle than they would get from the other player playing their punishment strategy, and therefore neither player has a utility-increasing deviation.

(e) Does this game have any dominant strategies? If so, which strategy is dominant?
Yes, player 1 has a dominant strategy of $A$.

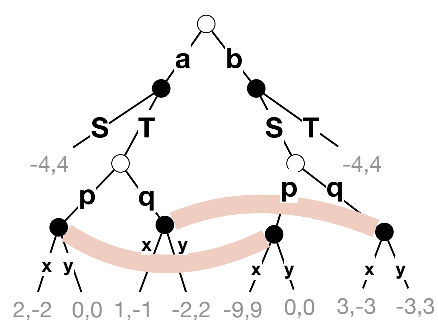
3. Consider the following imperfect information extensive form game:




   (a) What are the *information sets* for this game? (You can name *states* by the sequence of actions that lead to them).
   Player 1 (the white player) has information sets $\varnothing$, $\{aS\}$, and $\{bT, cP, cQ\}$, where $\varnothing$ is the root node.
   Player 2 (the black player) has information sets $\{a, b\}$ and $\{c\}$.

   (b) Is this a game of perfect or imperfect recall? Why or why not?
   This is a game of **imperfect recall**, because in information set $\{bT, cP, cQ\}$ the white player cannot remember whether they played $b$ or $c$.

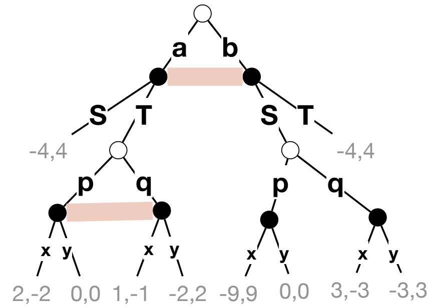
4. Consider the following imperfect information extensive form game:




   (a) What are the *information sets* for this game?
   Player 1 (the white player) has information sets $\varnothing$, $\{aT\}$, and $\{bS\}$.
   Player 2 (the black player) has information sets $\{a\}$, $\{b\}$, $\{aTp, bSp\}$, and $\{aTq, bSq\}$.

   (b) Is this a game of perfect or imperfect recall? Why or why not? This is a game of **imperfect recall**. In information set $\{aTp, bSp\}$, player 2 cannot tell whether player 1 played $a$ or $b$, even though they would have previously been in information set $\{a\}$ or $\{b\}$, and so would have known which of those two actions was played. The same forgetting happens in information set $\{aTq, bSq\}$. Furthermore, in those two information sets player 2 has also forgotten whether they themselves played $T$ or $S$. (For full marks it is sufficient to identify any single one of these memory failures).

5. Consider the following imperfect information extensive form game:




(a) What are the *information sets* for this game?
Player 1 has information sets $\varnothing$, $\{aT\}$, and $\{bS\}$.
Player 2 has information sets $\{a, b\}$, $\{aTp, aTq\}$, $\{bSp\}$, and $\{bSq\}$.

(b) Is this a game of perfect or imperfect recall? Why or why not?
This is a game of **perfect recall.** In every information set, the player is able to distinguish all the states that they were previously able to distinguish.

6. Consider the repeated game in which the Prisoner's Dilemma (below) is played twice.

| | $C$ | $D$ |
|---|---|---|
| $C$ | $-1, -1$ | $-5, 0$ |
| $D$ | $0, -5$ | $-3, -3$ |

Prisoner's Dilemma

(a) What are the *pure strategies* for row player?
The row player has the following strategies, where the first action shows what the player will do in the first game, and the second and third actions describe how the agent will play depending on what happened in the first game.

$$[C; (C, C) \mapsto C; (C, D) \mapsto C]$$
$$[C; (C, C) \mapsto C; (C, D) \mapsto D]$$
$$[C; (C, C) \mapsto D; (C, D) \mapsto C]$$
$$[C; (C, C) \mapsto D; (C, D) \mapsto D]$$
$$[D; (D, C) \mapsto C; (D, D) \mapsto C]$$
$$[D; (D, C) \mapsto C; (D, D) \mapsto D]$$
$$[D; (D, C) \mapsto D; (D, D) \mapsto C]$$
$$[D; (D, C) \mapsto D; (D, D) \mapsto D]$$

(b) Is there an equilibrium of this repeated game in which column ever plays $C$? Why or why not?

**No**, by the backward induction argument: in the last game, both players have a dominant strategy to play $D$. But then knowing that, the first game cannot affect the last game, and so both players have a dominant strategy to play $D$ in the first game as well.

7. Consider the following mechanism for allocating an item:

   - Every agent begins standing.
   - The auctioneer calls out prices in increasing order.
   - When the price is too high for a given agent, they sit down.
   - An agent cannot stand up again after they have sat.
   - When only a single agent remains standing, they have won the auction; they receive the item and are charged the last price that the auctioneer called.

   (a) Is this a *direct mechanism*?
   **No,** this is not a direct mechanism, because the agents' available actions (at each time step, either stand or sit) differ from a set of valuation reports. Note that this is true even though their actions do reveal information about their valuations.

   (b) Does this mechanism have a *dominant strategy* for the bidders? If so, what is it?
   **Yes.** An agent should remain standing so long as the current price is smaller than their valuation, regardless of what the other agents do.

   (c) Does this mechanism allocate the item to the bidder with the highest valuation in equilibrium? Why or why not?
   **Yes.** If every agent follows their dominant strategy, then the last agent standing will be the one with the highest valuation.

8. Consider the following mechanism for allocating an item:

   - Each secretly tells the auctioneer a number that they claim to be their valuation for the item.
   - The bidder who reports the largest valuation wins, and receives the item.
   - The winning bidder is charged $3\times$ the next-highest reported valuation.

   (a) Is this a *direct mechanism*?
   **Yes**; the actions available to each agent consist of explicitly reporting their valuation.

   (b) Is this a *truthful mechanism*? Why or why not?
   **No.** If every agent reports truthfully, and the highest valuation is not $3\times$ larger than the next highest valuation, then the highest bidder will be charged more than their value!

   As a concrete example, suppose every agent has a 50% chance of valuing the item at 1, and a 50% chance of valuing the item at 2. There if everyone bids truthfully, there are 3 possible cases:

      i. Winner bids 1, so does everyone else: winner will be charged 3, for a net utility of -2.
      ii. Winner bids 2, so does at least one other bidder: winner will be charged 6, for a net utility of -4.
      iii. Winner bids 2, everyone else bids 1: winner will be charged 3, for a net utility of -1.

   In every one of these cases, the winner would have been better off *not winning at all*; their best response to any profile of truthful bids is to untruthfully bid 0.

   (c) Does this mechanism have a *dominant strategy* for the bidders? If so, what is it?
   **No.** In the previous answer, the bidder had a single best response to every profile of *truthful* bids by the other bidders. However, if everyone else bids 0, then the best response is to bid something strictly greater than 0 (because then you get the item for free).