1. Consider the following 6×6 sliding tile position:

	3	2	1	4	5
6	7	8	9	10	11
12	13	14	15	16	17
18	19	20	21	22	23
24	25	26	27	28	29
30	31	32	33	34	35

   (a) How many *inversions* does this position have?
   3 inversions: $(3,2),(3,1),(2,1)$ are out of order, but $1,2,3$ are correctly ordered with respect to all later numbers, and all the later numbers are in the correct order.

   (b) Is this position *solvable*? Why or why not?
   Yes. Since there are an even number of columns, the position is solvable iff the number of inversions plus blank height is even. Blank height is 5 and there are 3 inversions, so the position is solvable because $5+3=8$ is even.

   (c) What is the *taxicab distance* for this position?
   3 needs to travel 1 step right; 2 needs to travel 1 step left; 1 needs to travel 3 steps left. $6,12,18,24,30$ all need to travel 5 right and 1 up. The remaining 27 numbers need to move 1 step left. So the taxicab distance is $1+1+3+5(6)+27(1)=62$.

   (d) How many *misplaced tiles* does this position have?
   All 35 tiles are misplaced.

2. Consider the following 3×3 sliding tile position:

8	6	7
2	5	4
3		1

   (a) How many *inversions* does this position have?
   The following pairs are out of order: $\{(8,6), (8,7), (8,2), (8,5), (8,4), (8,3), (8,1), (7,2), (7,5), (7,4), (7,3), (7,1), (6,2), (6,5), (6,4), (6,3), (6,1), (2,1), (5,4), (5,3), (5,1), (4,3), (4,1), (3,1)\}$. So there are 24 inversions.

   (b) Is this position *solvable*? Why or why not?
   Yes. There are an odd number of columns and an even number of inversions.

   (c) Give a non-vacuous (i.e., not 0) lower bound on the number of moves required to solve this position, and explain your reasoning.
   Each vertical move can reduce the number of inversions by at most 2 (and each horizontal move reduces the number of inversions by 0), so at least $24/2=12$ moves are necessary.

   Also acceptable: 7 of the tiles are misplaced, so at least 7 moves are necessary.

   Also acceptable: The taxicab distance is $3+2+4+2+0+2+4+4=21$, so at least 21 moves are necessary.

3. Are the following *admissible* heuristics for the sliding tile puzzle? Why or why not?

   (a) 2 times taxicab: Returns double the taxicab distance.

   No. Consider the following position:

1	2
	3

   . The taxicab distance is 1. This position can be solved with a single move, so double the taxicab distance (i.e., 2) is an over-estimate.

   (b) $\frac{1}{2}$ times taxicab: Returns half the taxicab distance.
   Yes. Taxicab distance is admissible, which means that it is never too large. But half the taxicab distance is always smaller than taxicab, so it is also never too large.

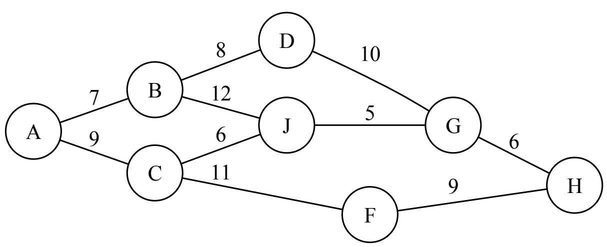
4. Which heuristic is more accurate: taxicab distance or number of misplacements? Explain why.
Taxicab distance is more accurate. Taxicab distance and misplacement count are both admissible, so they are never too large. Taxicab is always weakly more accurate, because the only way for an admissible heuristic to be inaccurate is to be too small, and misplacement count is always weakly smaller than taxicab because every misplacement must have a taxicab distance of at least 1.

5. For a general graph, under what circumstances is A* guaranteed to return:

   (a) The *cheapest* solution? When the heuristic is admissible.

   (b) The *shortest* solution? When every move has a cost of 1 (that is, when the cheapest path is guaranteed to also be the shortest).

6. Consider the following search graph. Edges are labelled with costs, and the table beneath gives the heuristic value for each node. The goal is to find a path from A to H. Recall that a node is *expanded* when it is removed from the fringe. List the order in which nodes are expanded by A*, and contents of the fringe after each expansion. When two nodes have identical priorities, expand the one that comes alphabetically first (i.e., break ties in alphabetical order).



n	A	B	C	D	J	F	G	H
$h(n)$	20	17	15	12	10	9	6	0

Node	Fringe
Expand A	B:24, C:24
Expand B	C:24, D:27, J:29
Expand C	D:27, J:25, F:29
Expand J	D:27, F:29, G:26
Expand G	D:27, F:29, H:26
Expand H (done)	D:27, F:29

7. For the same search graph, again breaking ties alphabetically:

   (a) What solution (i.e., path from A to H) will be found by *breadth-first search*? Explain why.
   $\langle A, C, F, H\rangle$, because it is the shortest solution, and breadth-first search always returns the shortest solution.

   (b) What solution (i.e., path from A to H) will be found by *depth-first search*? Explain why.
   $\langle A, B, D, G, H\rangle$, exploring each node's alphabetically-first neighbour in depth-first order.

   (c) In what sense is *random search* guaranteed to find a path to H?
   The only way for it to fail is for it to explore an infinitely-long path (e.g., by following cycles that don't involve H). But paths get less probable as they get longer (since each edge that gets added multiplies the total probability by the probability of the edge being chosen, which is less than 1). So the probability of a path approaches 0 as its length approaches infinity, which means that with probability 1 a finite-length path to H will be found.

   (d) In what sense is random search *not guaranteed* to find a path to H?
   It is possible for the search to explore a loopy path that does not include H forever, so in the worst case H will never be found.

8. Suppose that running breadth-first search on a 6×2 sliding tile puzzle takes 0.25s in the worst case (i.e., starting from an unsolvable position).

   (a) How many possible 6×2 positions are there? (You don't need to expand the expression to get an actual number) 12!

   (b) What is the average degree (i.e., number of neighbours) of a 6×2 position?
   Each position where the blank is in a corner has 2 neighbours, and every other position has 3 neighbours. So the average degree is $\frac{4}{12}(2) + \frac{8}{12}(3)12 = \frac{8}{3}$.

   (c) What is the average degree (i.e., number of neighbours) of a 3×4 position?
   The blank can be in one of 12 cells. 4 of those are corners with 2 neighbours; 6 of those are edges with 3 neighbours; and 2 of them are central cells with 4 neighbours. Diagramatically:

2	3	3	2
3	4	4	3
2	3	3	2

   So the average degree is
   $$\frac{4}{12}(2) + \frac{6}{12}(3) + \frac{2}{12}(4) = \frac{17}{6}.$$

   (d) Give an estimate of how long it would take to run breadth-first search on an unsolvable 3×4 position. Justify your answer.
   A 6×2 position takes 0.25s to run, and the search graph for a 3×4 position has
   $$\frac{12!17/6}{12!8/3} = \frac{17}{6} \cdot \frac{3}{8} = 1.0625$$
   times as many edges as the search graph for a 6×2 position, so the 3×4 position will take approximately $1.0625(0.25) = 0.265625s$ to run.

9. Recall that `stile/15puzzle.py` has three different subgoal schedules:

   (i) `[[1,2,3,4],[5,9,13],[6,7,8,10,11,12,14,15]]`

   (ii) `[[1],[2],[3,4],[5],[6],[7,8],[9,13],[10,14],[11,12,15]]`

   (iii) `[[1,2],[3,4],[5,6,7,8],[9,10,11,12,13,14,15]]`

   Schedule (i) places tiles $\{1, 2, 3, 4\}$ first, then tiles $\{5, 9, 13\}$; schedule (ii) places tile $\{1\}$ first, then tile $\{2\}$, etc. Suppose that we run `stile/15puzzle.py` on a solvable position.

   (a) How many positions are in the search graph for the first subgoal of each schedule?

   (i) There are $16!$ different positions of the full puzzle, but we actually only care about the positions of the 1,2,3,4, and blank tiles. Every ordering of the remaining 11 tiles is indistinguishable to us, so we divide by the $11!$ orderings of those tiles that are possible for each placement of the 1,2,3,4, and blank, giving

   $$\frac{16!}{11!} = 16 \times 15 \times 14 \times 13 \times 12 = 43,680$$

   distinguishable positions. (The ratio of factorials would be sufficient for full marks)

   (ii) Similarly, we only care about the placement of the 1 tile and the blank, so there are

   $$\frac{16!}{14!} = 16 \times 15 = 240$$

   distinguishable positions.

   (iii) Similarly, because we only care about the placement of the 1,2 and blank tiles, there are

   $$\frac{16!}{13!} = 16 \times 15 \times 14 = 3360$$

   distinguishable positions.

   (b) How many positions are in the search graph for the *second* subgoal of each schedule?

   (i) There are 12 unfrozen cells, each of which holds an unfrozen cell and a blank. We care about the placement of 3 tiles (5,9,13) plus the blank, so there are

   $$\frac{12!}{8!} = 12 \times 11 \times 10 \times 9 = 11,880$$

   distinguishable positions for this subgoal.

   (ii) There are 15 unfrozen cells, and we only care about the placement of one tile (the 2) and the blank, so there are

   $$\frac{15!}{13!} = 15 \times 14 = 210$$

   distinguishable positions.

   (iii) There are 14 unfrozen cells, and we care about the placement of two tiles plus the blank, so there are

   $$\frac{14!}{11!} = 14 \times 13 \times 12 = 2184$$

   distinguishable positions.

(c) Which schedule is likely to produce the *shortest solution*? Why?
Schedule (i), because it has a small number of large subgoals.

(d) Which schedule is likely to *search the fewest nodes*? Why?
Schedule (ii), because each subgoal is very small (and will therefore have a very small search space).

10. Suppose that `stile/15puzzle.py` is run on a solvable position, and successfully completes the first three subgoals of schedule (iii). Is the fourth subgoal guaranteed to be solvable on the resulting position? Why or why not?
Yes. Once the top two rows are solved, the only inversions will be between tiles in the bottom two rows. So the 2×4 puzzle represented by the bottom two rows has exactly the same number of inversions as the whole puzzle, which means that the bottom two rows are solvable if and only if the full puzzle is solvable. And since the starting position was solvable, the full puzzle must still be solvable.

11. Prove the following claim: For a 4×5 sliding tile puzzle (i.e., 4 rows, 5 columns), every move leaves the parity of the number of inversions unchanged.
Each horizontal move does not change the number of inversions at all. Each vertical move flips $5 - 1 = 4$ inversions, which means the number of inversions changes by an even number (4). But adding or subtracting an even number leaves the parity unchanged. □

12. Prove the following claim: There is no sequence of moves from a solvable sliding tile position to an unsolvable one.
Suppose for contradiction that there is! Let p be a solvable position, and u be an unsolvable position. If there is a path from p to u, then there is also a path from u to p. Since p is solvable, there must also be a path from p to the solved position. But that means that there is a path from u to the solved position (namely: go from u to p, and then from p to the solved position), which means that u wasn't unsolvable after all, and we have our contradiction. □

13. Consider the following 3×3 sliding tile position:

4	3	2
1	5	6
7	8	

(a) What *arrangement* does this position correspond to?
$4, 3, 2, 6, 5, 1, 7, 8$

(b) Give a sequence of moves that places the blank in the center square of the top row, without changing the arrangement.
8 right, 7 right, 1 down, 5 left, 6 left, 2 down, 3 right

(c) Give a sequence of moves that places 6 before 3 in the arrangement, while leaving the rest of the arrangement unchanged.

i. Move the blank to center top (as in the previous question):
8 right, 7 right, 1 down, 5 left, 6 left, 2 down, 3 right

ii. Then rotate in the elbow at top right: 6 up, 2 left, 3 down, 6 right

(The final 3 moves of 2 left, 3 down, 6 right are actually not necessary to satisfy the requirement; they just move the blank back to the starting position).

14. Here are two 3×7 sliding tile positions. Some of the numbers are hidden.

?	?	?	?	?	?	?
?	?	?	?	?	?	14
12	11	16	19	2	3	

?	?	?	?		4	15
7	22	11	24	18	?	?
?	?	?	?	?	?	?

(a) In the left position, what is the change in the number of inversions after the move **14 down**?
2 new inversions, 4 inversions removed; in total the number of inversions will be reduced by 2.

(b) In the right position, what is the change in the number of inversions after the move **18 up**?
4 new inversions, 2 inversions removed; in total the number of inversions will increase by 2.

(c) In the right position, what is the change in the number of inversions after the move **4 left**?
There will be no change in the number of inversions.