Perfect-Information Extensive Form Games

CMPUT 654: Modelling Human Strategic Behaviour

S&LB §5.1

Lecture Outline

1. Recap
2. Extensive Form Games
3. Subgame Perfect Equilibrium
4. Backward Induction

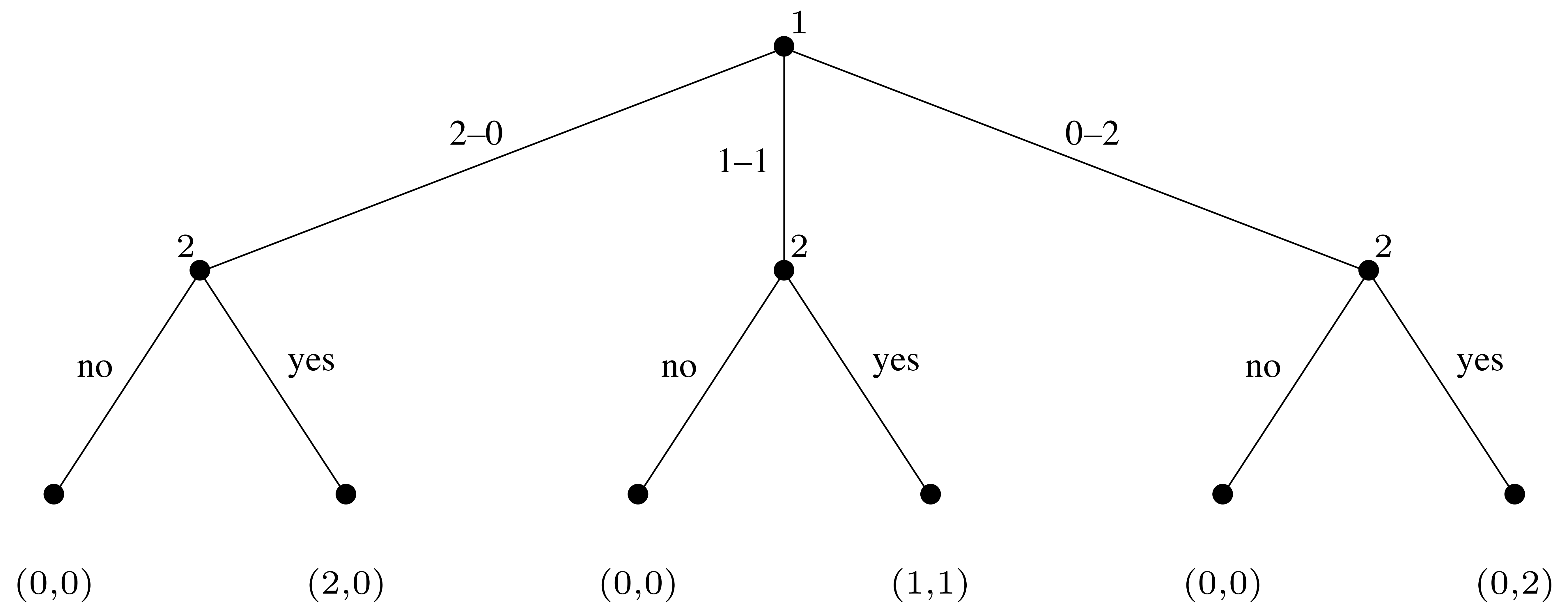
Recap

- **ε-Nash equilibria**: stable when agents have no deviation that gains them more than ε
- **Correlated equilibria**: stable when agents have **signals** from a possibly-correlated randomizing device
- **Linear programs** are a flexible encoding that can always be solved in **polytime**
- Finding a Nash equilibrium is **computationally hard** in general
- **Special cases** are efficiently computable:
    - Nash equilibria in zero-sum games
    - Maxmin strategies (and values) in two-player games
    - Correlated equilibrium

Extensive Form Games

- Normal form games don't have any notion of **sequence**: all actions happen **simultaneously**

- The **extensive form** is a game representation that explicitly includes temporal structure (i.e., a **game tree**)



Perfect Information

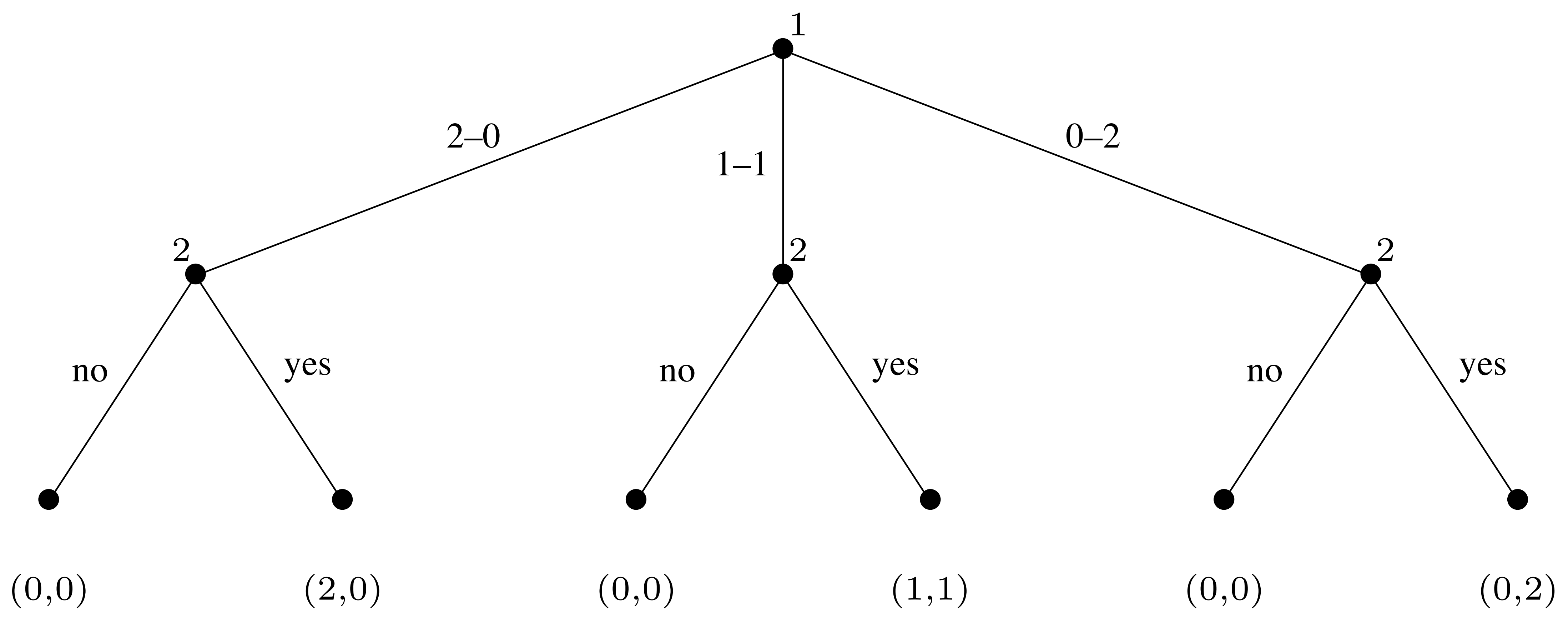
There are two kinds of extensive form game:

1. **Perfect information:** Every agent **sees all actions** of the other players (including Nature)
    - e.g.: Chess, checkers, Pandemic
    - This lecture!
2. **Imperfect information:** Some actions are **hidden**
    - Players may not know exactly where they are in the tree
    - e.g.: Poker, rummy, Scrabble

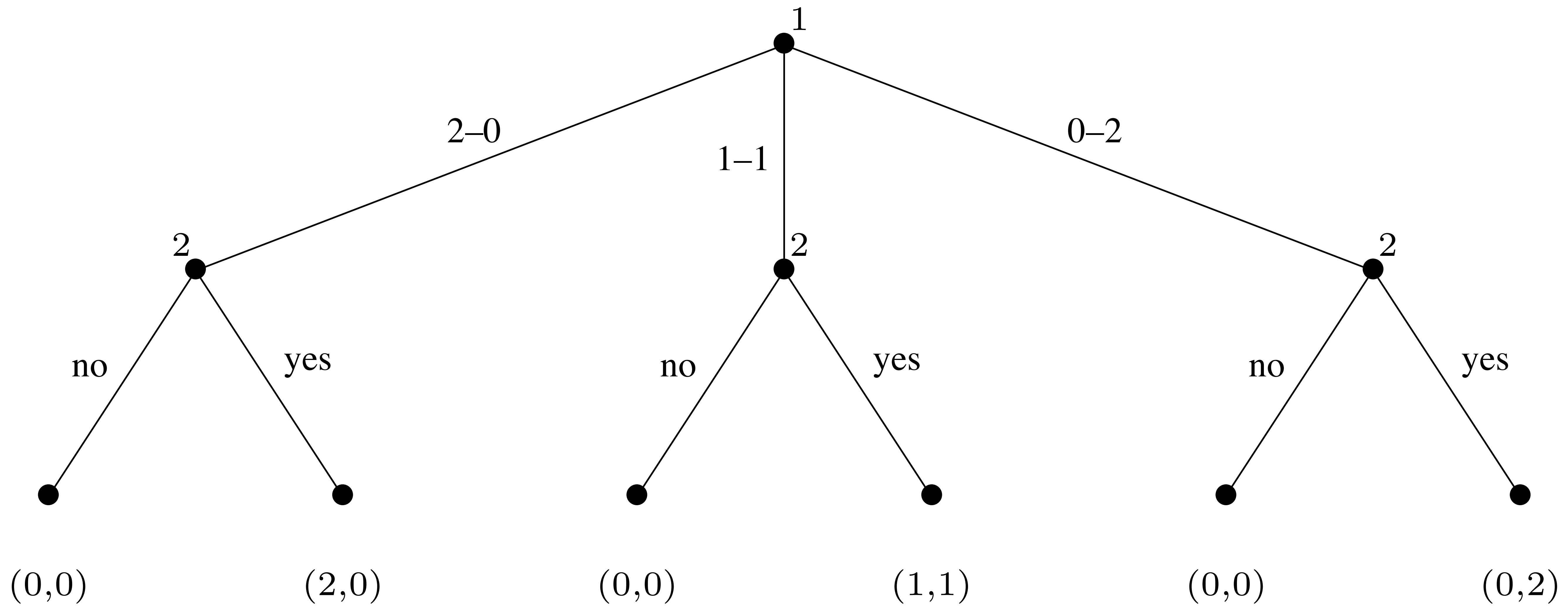
Perfect Information Extensive Form Game

Definition:
A **finite perfect-information game in extensive form** is a tuple $G = (N, A, H, Z, \chi, \rho, \sigma, u)$, where

- N is a set of n **players**,
- A is a single set of **actions**,
- H is a set of nonterminal **choice nodes**,
- Z is a set of **terminal nodes** (disjoint from H),
- $\chi : H \to 2^A$ is the **action function**,
- $\rho : H \to N$ is the **player function**,
- $\sigma : H \times A \to H \cup Z$ is the **successor function**,
- $u = (u_1, u_2, ..., u_n)$ is a **utility function** for each player $u_i : Z \to \mathbb{R}$.



Fun Game:
The Sharing Game



- Two siblings must decide how to share two $100 coins
- Sibling 1 suggests a division, then sibling 2 accepts or rejects
    - If rejected, nobody gets any coins.
- Play against 3 other people, once per person only

Pure Strategies

Question: What are the **pure strategies** in an extensive form game?

Definition:
Let $G = (N, A, H, Z, \chi, \rho, \sigma, u)$ be a perfect information game in extensive form. Then the **pure strategies of player *i*** consist of the cross product of actions available to player *i* at each of their choice nodes, i.e.,

$$\prod_{h \in H | \rho(h) = i} \chi(h)$$

- A pure strategy associates an action with **each** choice node, even those that will **never be reached**

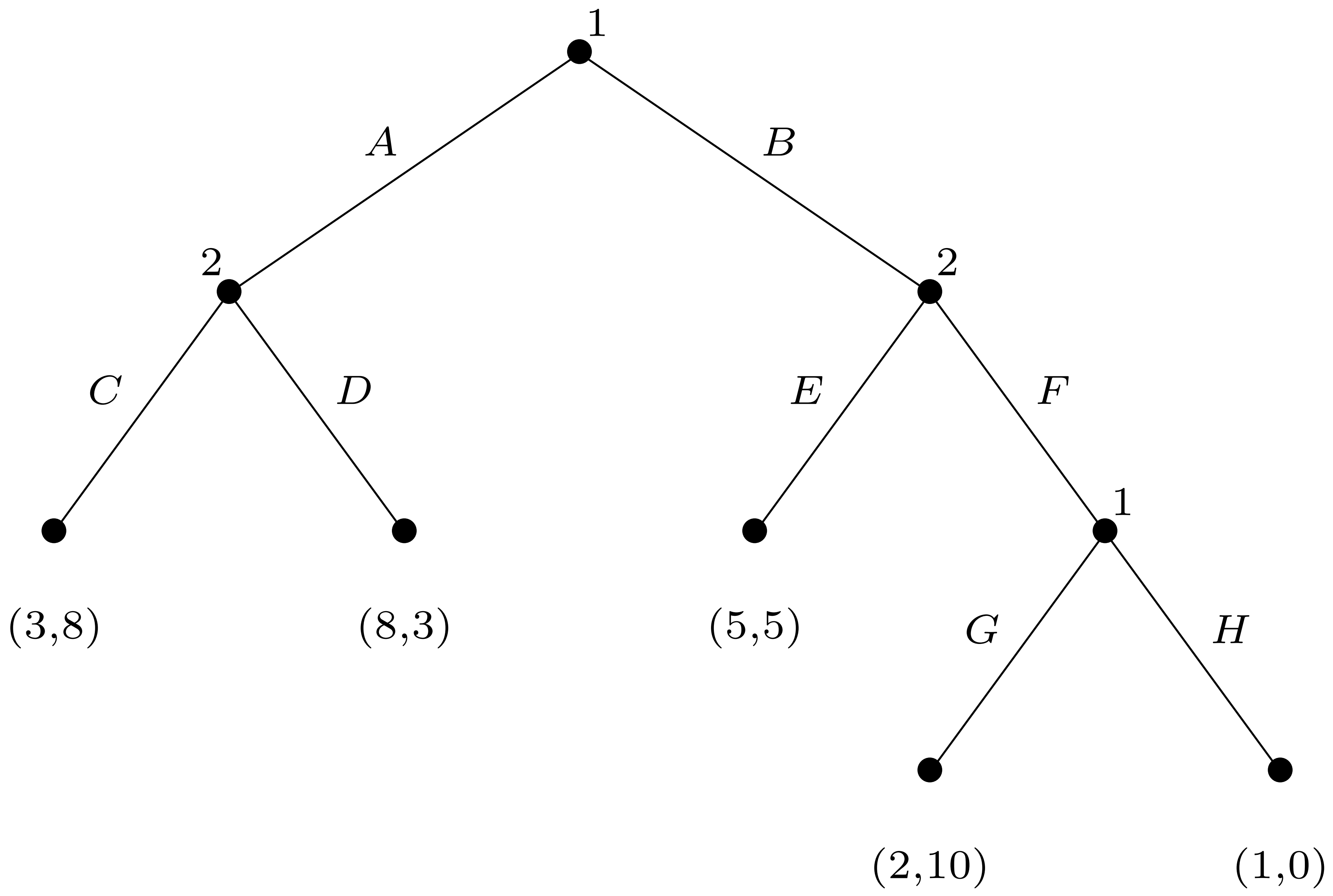
Pure Strategies Example

Question: What are the **pure strategies** for **player 2**?

- {(C,E), (C,F), (D,E), (D,F)}

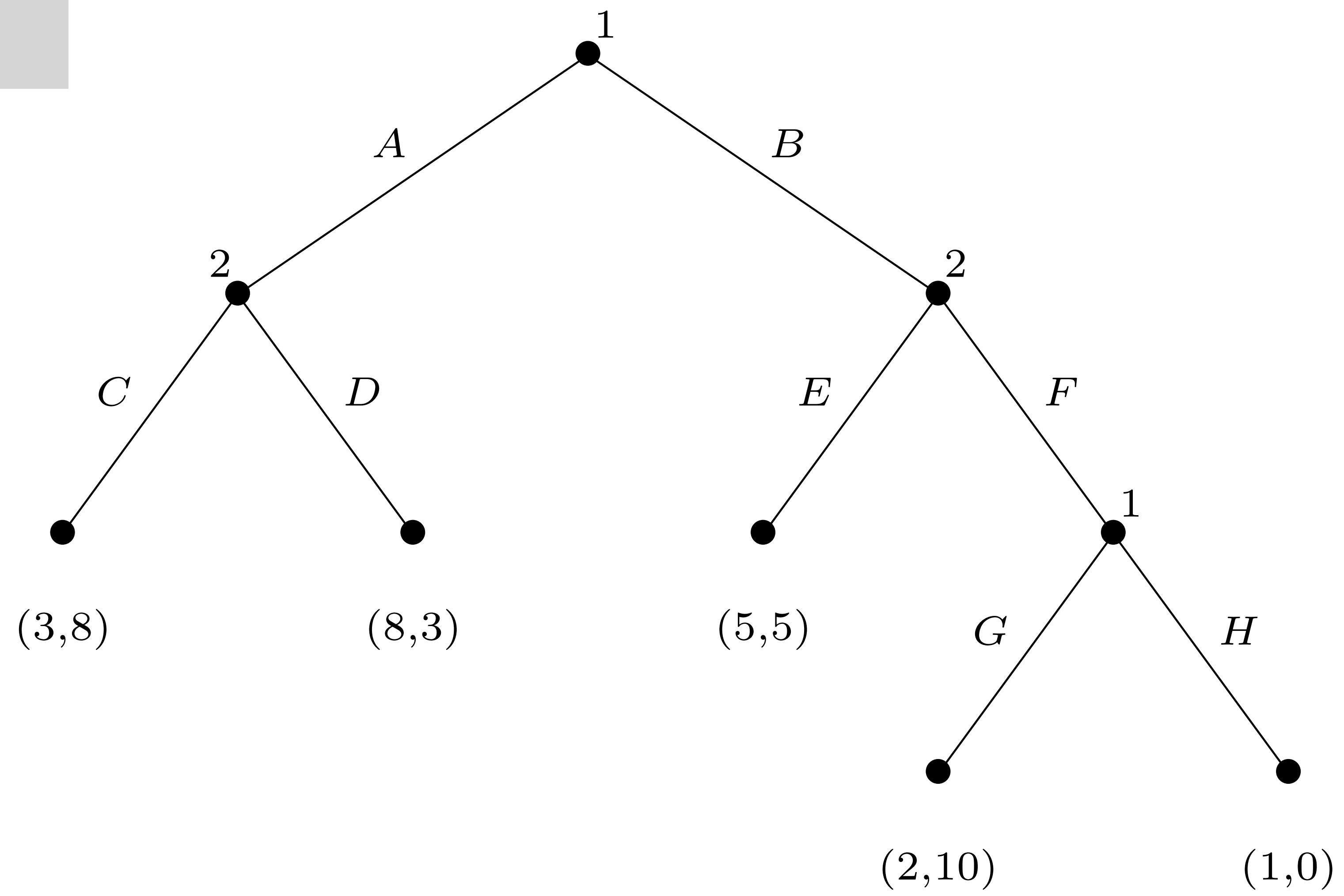
Question: What are the **pure strategies** for **player 1**?

- {(A,G), (A,H), (B,G), (G,H)}
- Note that these associate an action with the second choice node even when it can never be reached



Induced Normal Form

Question:
Which representation is more **compact**?



	C,E	C,F	D,E	D,F
A,G	3,8	3,8	8,3	8,3
A,H	3,8	3,8	8,3	8,3
B,G	5,5	2,10	5,5	2,10
B,H	5,5	1,0	5,5	1,0

- Any pair of pure strategies uniquely identifies a **terminal node**, which identifies a **utility** for each agent
- We have now defined a set of **agents**, **pure strategies**, and **utility functions**
- Any extensive form game defines a corresponding **induced normal form game**

Reusing Old Definitions

- We can plug our new definition of **pure strategy** into our existing definitions for:
  - Mixed strategy
  - Best response
  - Nash equilibrium (both pure and mixed strategy)

Question:

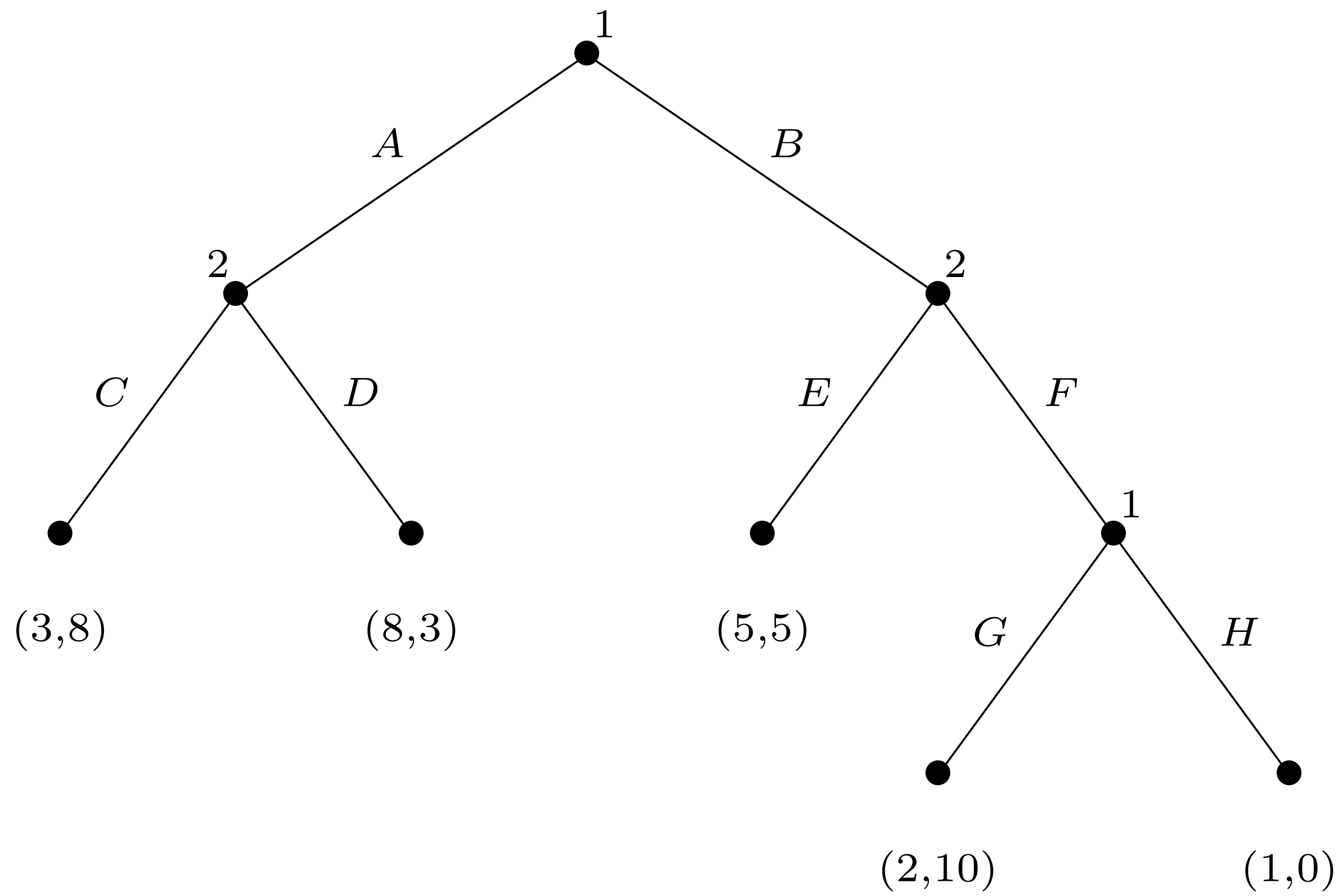
What is the definition of a **mixed strategy** in an extensive form game?

Pure Strategy Nash Equilibria

Theorem: [Zermelo 1913]
Every finite perfect-information game in extensive form has at least one **pure strategy Nash equilibrium**.

- Starting from the bottom of the tree, no agent needs to **randomize**, because they already know the best response
- There might be **multiple** pure strategy Nash equilibria in cases where an agent has multiple best responses at a **single choice node**

Pure Strategy Nash Equilibria



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- **Question:** What are the **pure-strategy Nash equilibria** of this game?
- **Question:** Do any of them seem **implausible**?

Subgame Perfection, informally

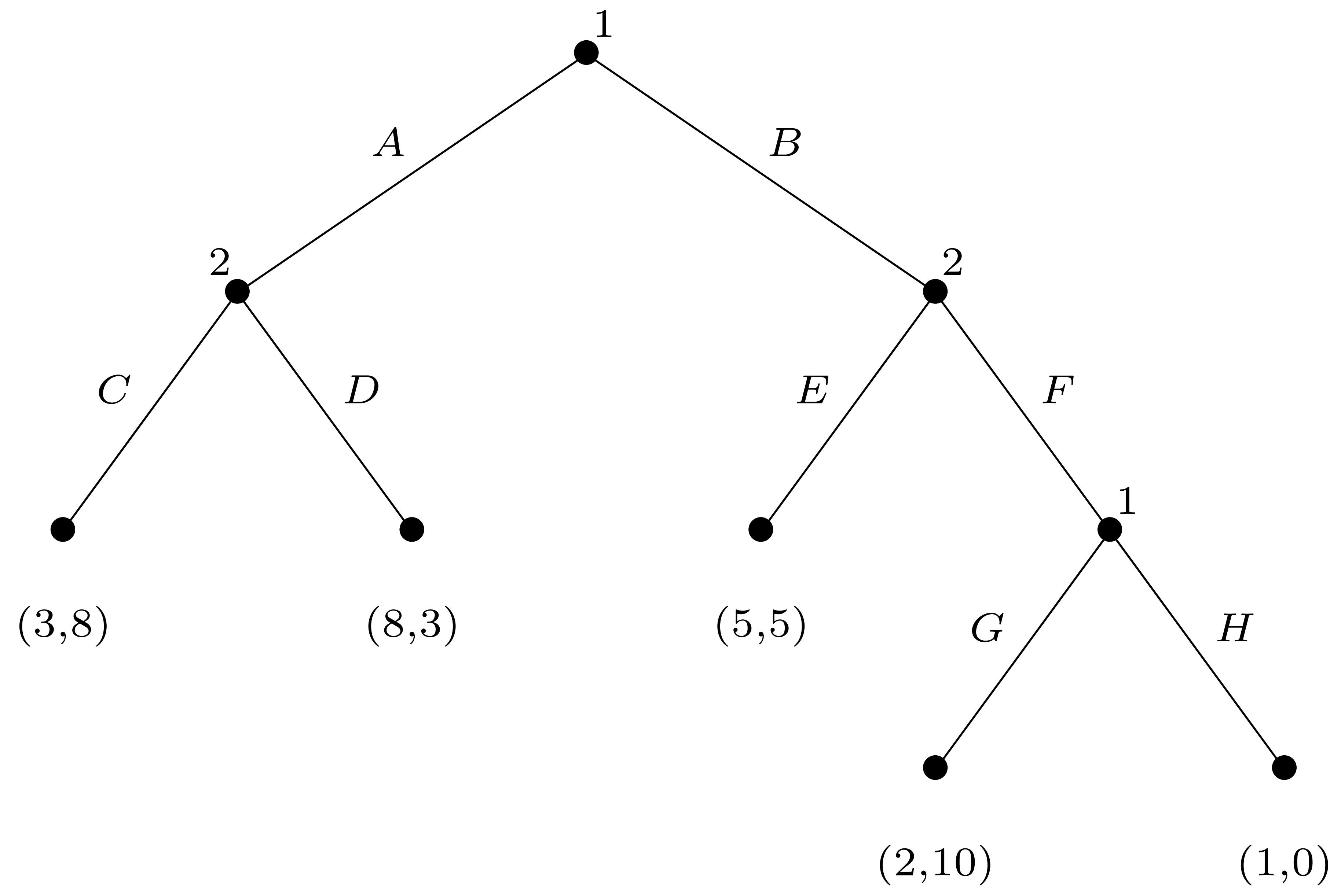
- Some equilibria seem less **plausible**

- (*BH,CE*): F has payoff 0 for player 2, because player 1 plays *H*, so their best response is to play *E*

    - But why would player 1 play *H* if **they got to that choice node**?

    - The equilibrium relies on a threat from player 1 that is not **credible**

- **Subgame perfect equilibria** are those that don't rely on non-credible threats

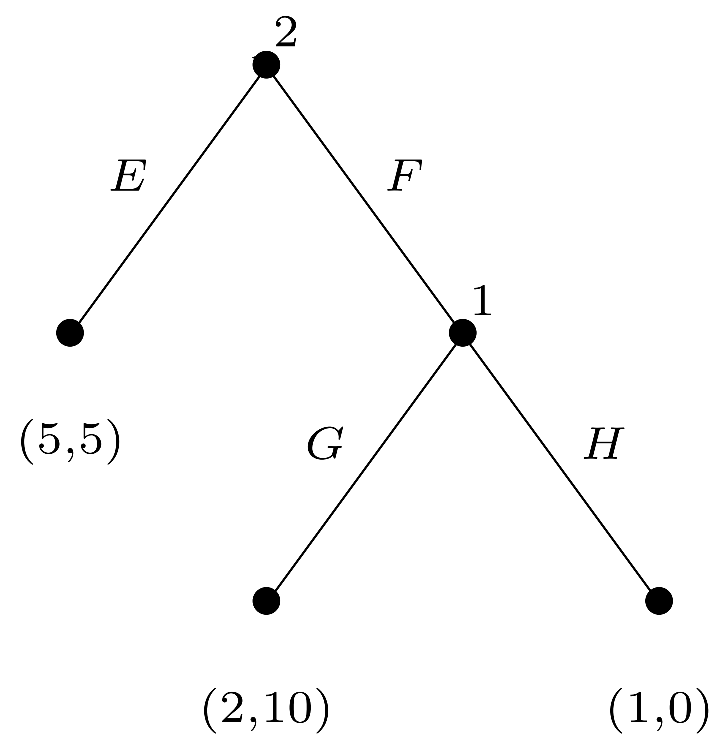


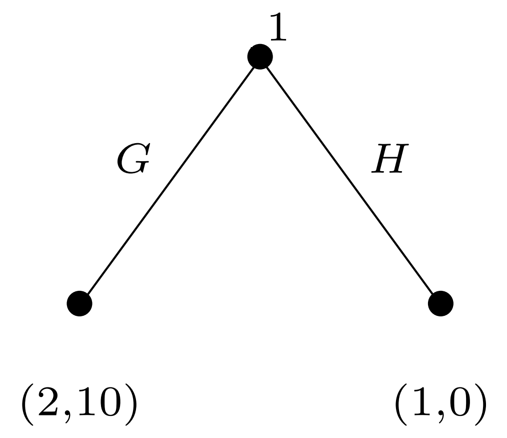
Subgames

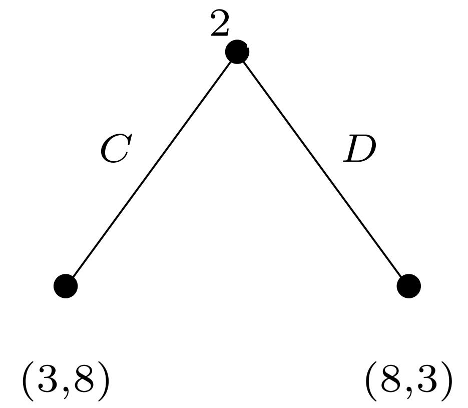
Definition:
The **subgame of *G* rooted at *h*** is the restriction of *G* to the descendants of *h*.

Definition:
The **subgames of *G*** are the subgames of *G* rooted at *h* for every choice node *h* ∈ *H*.

Examples:

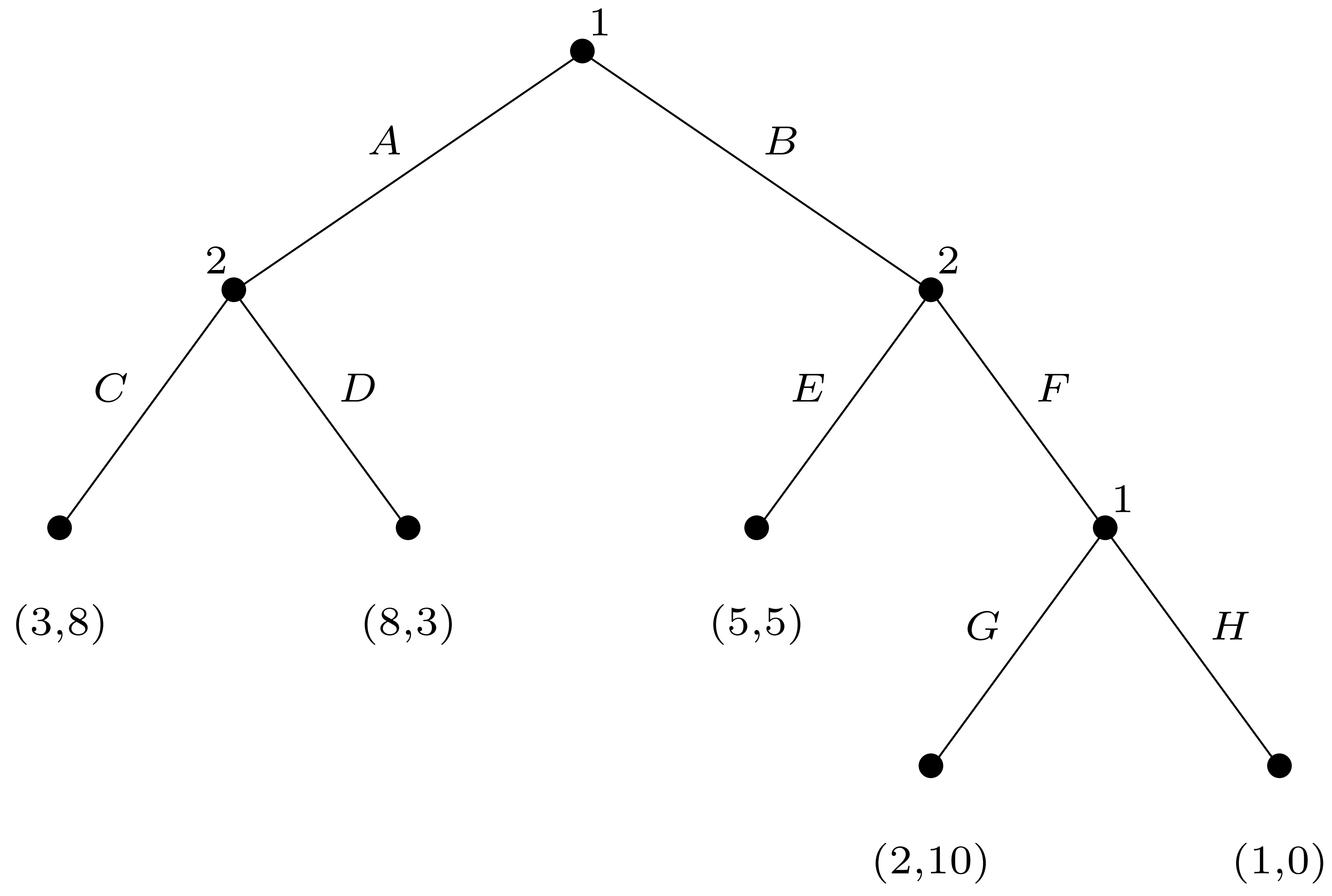






Subgame Perfect Equilibrium

Definition:
An strategy profile s is a **subgame perfect equilibrium** of *G* iff, for every subgame *G*' of *G*, the restriction of s to G' is a Nash equilibrium of *G*'.



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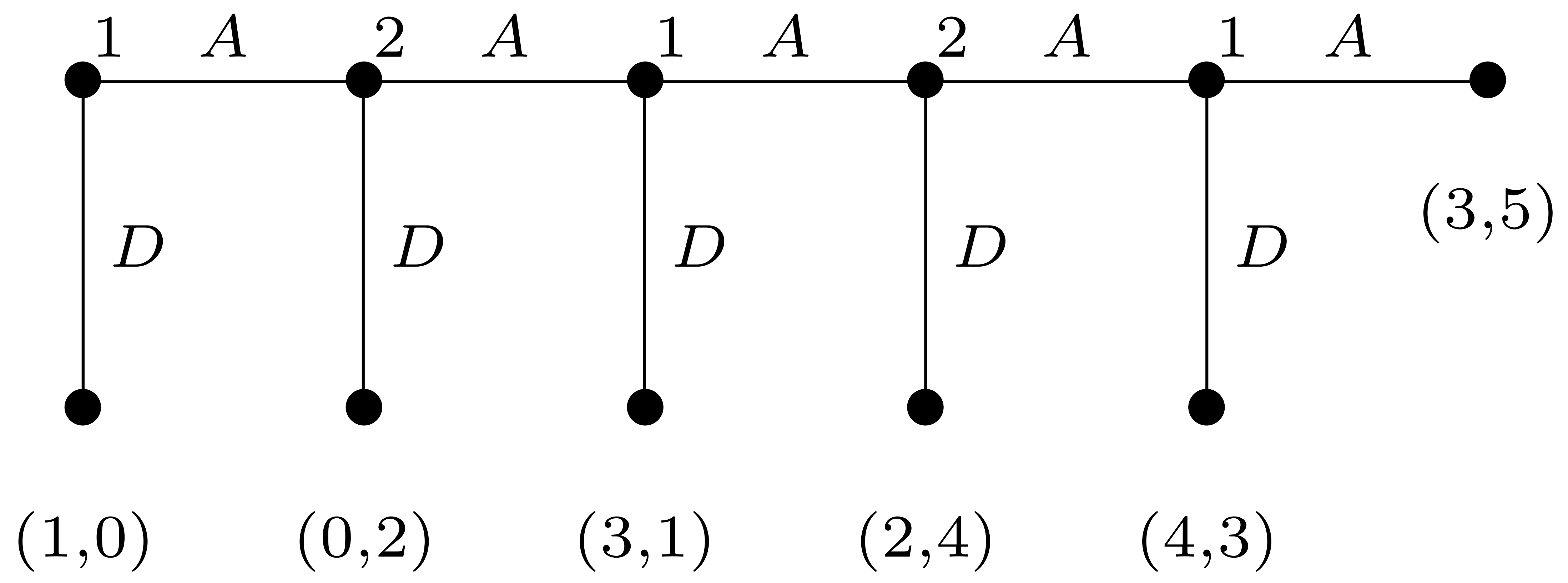
Backward Induction

- **Backward induction** is a straightforward algorithm that is guaranteed to compute a subgame perfect equilibrium
- **Idea:** Replace subgames lower in the tree with their equilibrium values

BACKWARDINDUCTION(h):
    if h is terminal:
        return $u(h)$
    $i := \rho(h)$
    $U := -\infty$
    for each h' **in** $\chi(h)$:
        V = BACKWARDINDUCTION(h')
        if $V_i > U_i$:
            $U_i := V_i$
    return U

Fun Game: Centipede

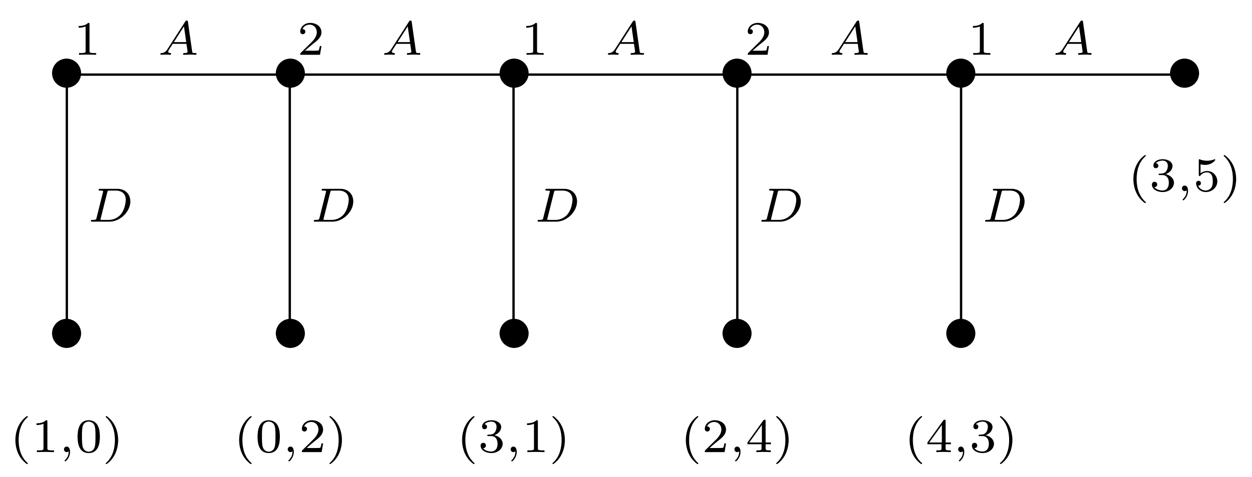


Question:

What is the unique **subgame perfect equilibrium** for Centipede?

- At each stage, one of the players can go **Across** or **Down**
- If they go Down, the game ends.
- Play against four people! Try to play each role at least once.

Backward Induction Criticism



- The **unique** subgame perfect equilibrium is for each player to go Down **at the first opportunity**
- **Empirically**, this is not how real people tend to play!
- **Theoretically**, what should you do if you arrive at an **off-path** node?
    - How do you update your beliefs to account for this probability 0 event?
    - If player 1 knows that you will update your beliefs in a way that causes you not to go down, then going down is no longer their only rational choice...

Summary

- **Extensive form games** allow us to represent sequential action
  - **Perfect information**: when we see everything that happens
- **Pure strategies** for extensive form games map **choice nodes** to **actions**
  - **Induced normal form** is the normal form game with these pure strategies
  - Notions of mixed strategy, best response, etc. translate directly
- **Subgame perfect equilibria** are those which do not rely on non-credible threats
  - Can always find a subgame perfect equilibrium using **backward induction**
  - But backward induction is theoretically and practically **complicated**