Further
Solution Concepts

CMPUT 654: Modelling Human Strategic Behaviour

S&LB §3.4

Lecture Outline

1. Recap
2. Maxmin Strategies
3. Dominated Strategies
4. Rationalizability

Recap: Pareto Optimality

Definition: Outcome *o* **Pareto dominates** *o*' if

1. $\forall i \in N : o \succeq_i o'$, and

2. $\exists i \in N : o \succ_i o'$.

Equivalently, **action profile** *a* Pareto dominates *a*' if $u_i(a) \geq$ ui(a') for all *i* and $u_i(a) > u_i(a')$ for some *i*.

Definition: An outcome o^* is **Pareto optimal** if no other outcome Pareto dominates it.

Recap: Best Response and Nash Equilibrium

Definition:
The set of *i*'s **best responses** to a strategy profile $s \in S$ is

$$BR_i(s_{-i}) \doteq \{s_i^* \in S \mid u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i}) \ \ \forall s_i \in S_i\}$$

Definition:
A strategy profile $s \in S$ is a **Nash equilibrium** iff

$$\forall i \in N, \ \ s_i \in BR_{-i}(s_{-i})$$

- When at least one s_i is mixed, s is a **mixed strategy Nash equilibrium**

Maxmin Strategies

What is the maximum amount that an agent can **guarantee** themselves in expectation?

Definition:
A **maxmin strategy** for *i* is a strategy $\bar{s}_i$ that maximizes *i*'s worst-case payoff:

$$\bar{s}_i = \arg\max_{s_i \in S_i} \left[\min_{s_{-i} \in S_i} u_i(s_i, s_{-i}) \right]$$

Question:

Why would an agent want to play a maxmin strategy?

Definition:
The **maxmin value** of a game for i is the value $\bar{v}_i$ guaranteed by a maxmin strategy:

$$\bar{v}_i = \max_{s_i \in S_i} \left[\min_{s_{-i} \in S_i} u_i(s_i, s_{-i}) \right]$$

Minmax Strategies

The corresponding strategy for the other player is the minmax strategy: the strategy that minimizes the other player's payoff.

Definition: (two-player games)
In a two-player game, the **minmax strategy** for player i against player -i is

$$\underline{s}_i = \arg\min_{s_i \in S_i} \left[\max_{s_{-i} \in S_{-i}} u_{-i}(s_i, s_{-i}) \right].$$

Definition: (*n*-player games)
In an *n*-player game, the **minmax strategy** for player *i* against player *j* ≠ *i* is *i*'s component of the mixed strategy profile $\underline{s}_{-j}$ in the expression

$$\underline{s}_{-j} = \arg\min_{s_{-j} \in S_{-j}} \left[\max_{s_j \in S_j} u_j(s_j, s_{-j}) \right],$$

and the **minmax value** for player *j* is $\underline{v}_j = \min_{s_{-j} \in S_{-j}} \max_{s_j \in S_j} u_j(s_j, s_{-j})$.

Question:

Why would an agent want to play a maxmin strategy?

Minimax Theorem

Theorem: [von Neumann, 1928]
In any finite, two-player, zero-sum game, in any Nash equilibrium, each player receives an expected utility v_i equal to both their maxmin and their minmax value.

Proof sketch:

1. Suppose that $v_i < \bar{v}_i$. But then i could guarantee a higher payoff by playing their maxmin strategy. So $v_i \geq \bar{v}_i$.
2. -i's equilibrium payoff is $v_{-i} = \max_{s_{-i}} u_{-i}(s_i^*, s_{-i})$
3. Equivalently, $v_i = \min_{s_{-i}} u_i(s_i^*, s_{-i})$, since the game is zero sum.
4. So $v_i = \min_{s_{-i}} u_i(s_i^*, s_{-i}) \leq \max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i}) = \bar{v}_i$. ∎

Minimax Theorem Implications

1. Each player's maxmin value is equal to their minmax value. We call this the **value of the game**.

2. For both players, the maxmin strategies and the Nash equilibrium strategies are the same sets.

3. Any **maxmin strategy profile** (a profile in which both agents are playing maxmin strategies) is a Nash equilibrium. Therefore, each player gets the same payoff in every Nash equilibrium (namely, their value for the game).

Dominated Strategies

When can we say that one strategy is **definitely** better than another, from an **individual's** point of view?

Definition: (domination)
Let s_i, $s_i' \in S_i$ be two of player i's strategies. Then

1. s_i **strictly dominates** s_i' if $\forall s_{-i} \in S_{-i} : u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$.

2. s_i **weakly dominates** s_i' if $\forall s_{-i} \in S_{-i} : u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$ and $\exists s_{-i} \in S_{-i} : u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$.

3. s_i **very weakly dominates** s_i' if $\forall s_{-i} \in S_{-i} : u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$.

Dominant Strategies

Definition:
A strategy is (strictly, weakly, very weakly) **dominant** if it (strictly, weakly, very weakly) dominates every other strategy.

Definition:
A strategy is (strictly, weakly, very weakly) **dominated** if is is (strictly, weakly, very weakly) dominated by some other strategy.

Definition:
A strategy profile in which every agent plays a (strictly, weakly, very weakly) dominant strategy is an **equilibrium in dominant strategies**.

Questions:

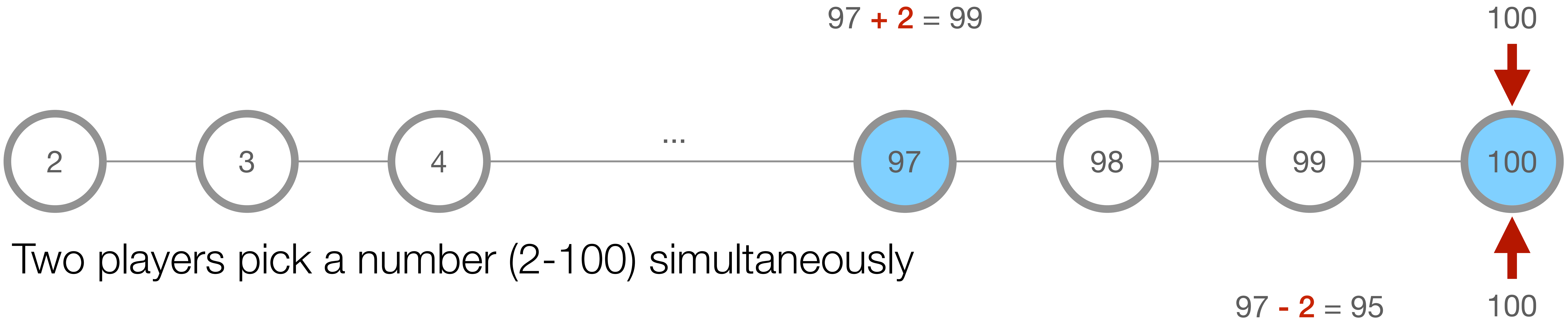
1. Are dominant strategies guaranteed to exist?
2. What is the maximum number of **weakly dominant** strategies?
3. Is an equilibrium in dominant strategies also a Nash equilibrium?

Prisoner's Dilemma again

	Coop.	Defect
Coop.	-1,-1	-5,0
Defect	0,-5	-3,-3

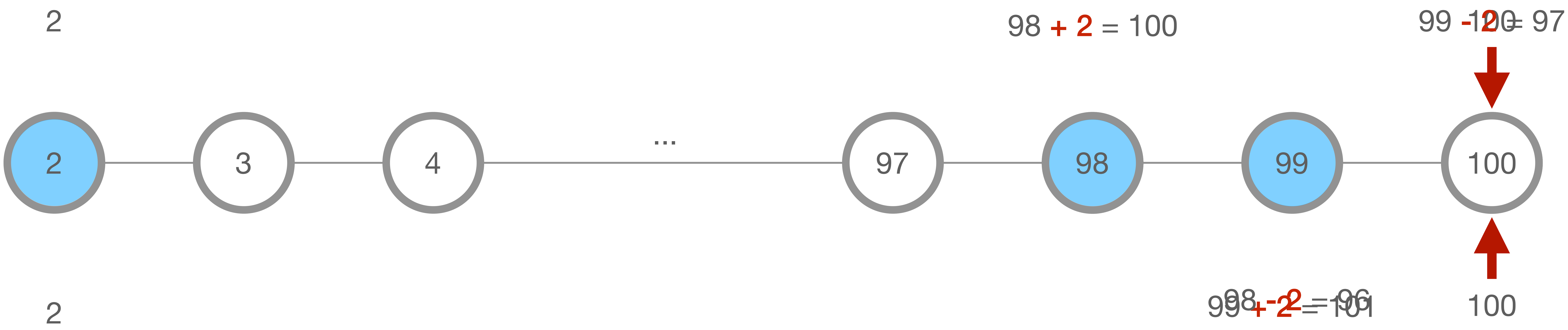
- Defect is a strictly dominant pure strategy in Prisoner's Dilemma.
- **Question:** Why would an agent want to play a **dominant** strategy?
- **Question:** Why would an agent want to play a **dominated** strategy?

Fun Game:
Traveller's Dilemma



- Two players pick a number (2-100) simultaneously
- If they pick the same number *x*, then they both get $*x* payoff
- If they pick different numbers:
    - Player who picked lower number gets **lower** number, plus **bonus** of $2
    - Player who picked higher number gets **lower** number, minus **penalty** of $2
- Play against someone near you, three times in total. Keep track of your payoffs!

Traveller's Dilemma



- Traveller's Dilemma has a unique Nash equilibrium

Iterated Removal of Dominated Strategies

- No **strictly dominated** pure strategy will ever be played by a fully rational agent.
- So we can remove them, and the game remains **strategically equivalent**
- But! Once you've removed a dominated strategy, another strategy that wasn't dominated before might **become dominated** in the new game.
    - It's safe to remove this newly-dominated action, because it's never a best response *to an action that the opponent would ever play*.
- You can repeat this process until there are no dominated actions left

Iterated Removal of Dominated Strategies

- Removing **strictly dominated** strategies preserves **all equilibria** (**Why?**)
- Removing weakly or very weakly dominated strategies preserves **at least one equilibrium**. (**Why?**)
    - But because not all equilibria are necessarily preserved, the order in which strategies are removed can matter.

Rationalizability

- We saw in the utility theory lecture that **beliefs** need not be **objective** (or accurate)
- What strategies could possibly be played by:
    1. A **rational** player...
    2. ...with **common knowledge** of the rationality of **all players?**
- Any strategy that is a **best response to some beliefs** consistent with these two conditions is **rationalizable**.

Questions:

1. What kind of strategy definitely could **not** be played by a rational player with common knowledge of rationality?
2. Is a rationalizable strategy guaranteed to exist?
3. Can a game have more than one rationalizable strategy?

Summary

- **Maxmin strategies** maximize an agent's **guaranteed payoff**
- **Minmax strategies** minimize the other agent's payoff as much as possible
- The **Minimax Theorem**:
    - Maxmin and minmax strategies are the **only** Nash equilibrium strategies in **zero-sum games**
    - Every Nash equilibrium in a zero-sum game has the **same payoff**
- **Dominated strategies** can be removed **iteratively** without strategically changing the game (too much)
- **Rationalizable** strategies are any that are a **best response** to some **rational belief**