Imperfect Information
Extensive Form Games

CMPUT 654: Modelling Human Strategic Behaviour

S&LB §5.2-5.2.2
Assignment Hint: Mixed Strategy Nash by Hand

- Recall that if we know the support of an equilibrium in a two-player game we can compute its equilibrium with an LP.

- For small games, you can just solve a system of equations for the probabilities of each action by hand.

**Key points:**

1. If player $i$ is mixing between two strategies in equilibrium, then they must **both** be best responses.

$$\sum_{a_{-i} \in \sigma_{-i}} s_{-i}(a_{-i}) u_i(a_i, a_{-i}) = v_i \quad \forall i \in \{1,2\}, a_i \in \sigma_i$$

$$\sum_{a_{-i} \in \sigma_{-i}} s_{-i}(a_{-i}) u_i(a_i, a_{-i}) \leq v_i \quad \forall i \in \{1,2\}, a_i \not\in \sigma_i$$

$$s_i(a_i) \geq 0 \quad \forall i \in \{1,2\}, a_i \in \sigma_i$$

$$s_i(a_i) = 0 \quad \forall i \in \{1,2\}, a_i \not\in \sigma_i$$

$$\sum_{a_i \in A_i} s_i(a_i) = 1 \quad \forall i \in \{1,2\}$$

2. Whether two strategies are best responses for $i$ depends upon the probabilities that the other player plays their strategies.

$$\sum_{a_i \in \sigma_i} s_i(a_i) = 1 \quad \forall i \in \{1,2\}$$
Recap: Perfect Information
Extensive Form Game

Definition:
A finite perfect-information game in extensive form is a tuple $G = (N, A, H, Z, \chi, \rho, \sigma, u)$, where

- $N$ is a set of $n$ players,
- $A$ is a single set of actions,
- $H$ is a set of nonterminal choice nodes,
- $Z$ is a set of terminal nodes (disjoint from $H$),
- $\chi : H \rightarrow 2^A$ is the action function,
- $\rho : H \rightarrow N$ is the player function,
- $\sigma : H \times A \rightarrow H \cup Z$ is the successor function,
- $u = (u_1, u_2, \ldots, u_n)$ is a profile of utility functions for each player, with $u_i : Z \rightarrow \mathbb{R}$.
Recap: Pure Strategies

Definition:
Let $G = (N, A, H, Z, \chi, \rho, \sigma, u)$ be a perfect information game in extensive form. Then the pure strategies of player $i$ consist of the cross product of actions available to player $i$ at each of their choice nodes, i.e.,

$$
\prod_{h \in H | \rho(h) = i} \chi(h).
$$

Note: A pure strategy associates an action with each choice node, even those that will never be reached.
Recap: Induced Normal Form

- Any pair of pure strategies uniquely identifies a **terminal node**, which identifies a **utility** for each agent.
- We have now defined a set of **agents**, **pure strategies**, and **utility functions**.
- Any extensive form game defines a corresponding **induced normal form game**.
Recap: Backward Induction

- **Backward induction** is a straightforward algorithm that is guaranteed to compute a subgame perfect equilibrium.

- **Idea**: Replace subgames lower in the tree with their equilibrium values.
Lecture Outline

1. Hints & Recap
2. Imperfect Information Games
3. Behavioural vs. Mixed Strategies
4. Perfect vs. Imperfect Recall
5. Computational Issues
Imperfect Information, informally

- **Perfect information** games model **sequential** actions that are **observed** by all players
  - **Randomness** can be modelled by a special *Nature* player with constant utility

- But many games involve **hidden** actions
  - Cribbage, poker, Scrabble
  - Sometimes actions of the **players** are hidden, sometimes *Nature's* actions are hidden, sometimes both

- **Imperfect information extensive form games** are a model of games with sequential actions, some of which may be **hidden**
Imperfect Information Extensive Form Game

**Definition:**
An imperfect information game in extensive form is a tuple $G = (N, A, H, Z, \chi, \rho, \sigma, u, I)$, where

- $(N, A, H, Z, \chi, \rho, \sigma, u)$ is a perfect information extensive form game, and

- $I = (I_1, \ldots, I_n)$, where $I_i = (I_{i,1}, \ldots, I_{i,k_i})$ is an equivalence relation on (i.e., partition of) $\{h \in H : \rho(h) = i\}$ with the property that $\chi(h) = \chi(h')$ and $\rho(h) = \rho(h')$ whenever there exists a $j$ for which $h \in I_{i,j}$ and $h' \in I_{i,j}$.
Imperfect Information
Extensive Form Example

- The elements of the partition are sometimes called **information sets**
- Players **cannot distinguish** which **history** they are in within an information set
- **Question:** What are the information sets for each player in this game?
Pure Strategies

**Question:** What are the pure strategies in an imperfect information extensive form game?

**Definition:**
Let $G = (N, A, H, Z, \chi, \rho, \sigma, u, I)$ be an imperfect information game in extensive form. Then the pure strategies of player $i$ consist of the cross product of actions available to player $i$ at each of their information sets, i.e.,

$$\prod_{I_i,j \in I_i} \chi(h)$$

- A pure strategy associates an action with each information set, even those that will never be reached.

**Questions:**
In an imperfect information game:
1. What are the mixed strategies?
2. What is a best response?
3. What is a Nash equilibrium?
Induced Normal Form

- Any pair of pure strategies uniquely identifies a terminal node, which identifies a utility for each agent.

- We have now defined a set of agents, pure strategies, and utility functions.

- Any extensive form game defines a corresponding induced normal form game.

**Question:** Can you represent an arbitrary perfect information extensive form game as an imperfect information extensive form game?
Normal to Extensive Form

• Unlike perfect information games, we can go in the opposite direction and represent any normal form game as an imperfect information extensive form game.

• Players can play in any order (why?)

• Question: What happens if we run this translation on the induced normal form?
Behavourial vs. Mixed Strategies

Definition:
A **mixed strategy** $s_i \in \Delta(A^I_i)$ is any distribution over an agent's **pure strategies**.

Definition:
A **behavioural strategy** $b_i \in [\Delta(A)]^I_i$ is a probability distribution over an agent's actions at an **information set**, which is **sampled independently** each time the agent arrives at the information set.
Behavioural vs. Mixed Example

- **Behavioural strategy**: ([.6:A, .4:B], [.6:G, .4:H])
- **Mixed strategy**: [.6:(A,G), .4:(B,H)]
- **Question**: Are these strategies equivalent? (why?)
- **Question**: Can you construct a mixed strategy that is equivalent to the behavioural strategy above?
- **Question**: Can you construct a behavioural strategy that is equivalent to the mixed strategy above?
Perfect Recall

Definition:
Player $i$ has **perfect recall** in an imperfect information game $G$ if for any two nodes $h, h'$ that are in the same information set for player $i$, for any path $h_0, a_0, h_1, a_1, \ldots, h_n, h$ from the root of the game to $h$, and for any path $h_0, a'_0, h'_1, a'_1, \ldots, h'_m, h'$ from the root of the game to $h'$, it must be the case that:

1. $n = m$, and

2. for all $0 \leq j \leq n$, if $\rho(h_j) = 1$, then $h_j$ and $h'_j$ are in the same information set, and

3. for all $0 \leq j \leq n$, if $\rho(h_j) = 1$, then $a_j = a'_j$.

$G$ is a **game of perfect recall** if every player has perfect recall in $G$. 
Perfect Recall Examples

Question: Which of the above games is a game of perfect recall?
Imperfect Recall Example

- Player 1 doesn't remember whether they have played L before or not. In this case, that is because they visit the same information set multiple times.

- **Question:** Can you construct a mixed strategy equivalent to the behavioural strategy [.5:L, .5R]?

- **Question:** Can you construct a behavioural strategy equivalent to the mixed strategy [.5:L, .5:R]?

- **Question:** What is the mixed strategy equilibrium in this game?

- **Question:** What is an equilibrium in behavioural strategies in this game?
Imperfect Recall Applications

**Question:** When is it **useful** to model a scenario as a game of **imperfect recall**?

1. When the **actual agents** being modelled may **forget** previous history
   - Including cases where the agents strategies really are executed by **proxies**

2. As an **approximation technique**
   - E.g., **poker:** The exact cards that have been played to this point may not matter as much as some coarse grouping of which cards have been played
     - Grouping the cards into equivalence classes is a **lossy** approximation
Kuhn's Theorem

**Theorem:** [Kuhn, 1953]
In a game of perfect recall, any mixed strategy of a given agent can be replaced by an equivalent behavioural strategy, and any behavioural strategy can be replaced by an equivalent mixed strategy.

- Here, two strategies are equivalent when they induce the same probabilities on outcomes, for any fixed strategy profile (mixed or behavioural) of the other agents.

**Corollary:**
Restricting attention to behavioural strategies does not change the set of Nash equilibria in a game of perfect recall. (why?)
Computational Issues

- **Question:** Can we use *backward induction* to find an equilibrium in an imperfect information extensive form game?

- We can just use the *induced normal form* to find the equilibrium of any imperfect information game
  - But the induced normal form is *exponentially larger* than the extensive form

- Can use the *sequence form* [S&LB §5.2.3] in games of *perfect recall*:
  - **Zero-sum games:** *polynomial* in size of extensive form (i.e., exponentially faster than LP formulation on normal form)
  - **General-sum games:** *exponential* in size of extensive form (i.e., exponentially faster than converting to normal form)
Summary

• **Imperfect information extensive form games** are a model of games with sequential actions, some of which may be hidden
  
  • Histories are partitioned into **information sets**
  
  • Player **cannot distinguish** between histories in the same information set
  
• **Pure strategies** map each information set to an action
  
  • **Mixed strategies** are distributions over pure strategies
  
  • **Behavioural strategies** map each information set to a distribution over actions
  
  • In games of perfect recall, mixed strategies and behavioural strategies are **interchangeable**

• A player has **perfect recall** if they **never forget** anything they knew about actions so far