Perfect-Information Extensive Form Games

CMPUT 654: Modelling Human Strategic Behaviour

S&LB §5.1

Recap: Best Response and Nash Equilibrium

Definition:
The set of i's **best responses** to a strategy profile $s_{-i} \in S_{-i}$ is

$$BR_i(s_{-i}) \doteq \{s_i^* \in S \mid u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i}) \;\; \forall s_i \in S_i\}$$

Definition:
A strategy profile $s \in S$ is a **Nash equilibrium** iff

$$\forall i \in N : \quad s_i \in BR_{-i}(s_{-i})$$

- When at least one s_i is mixed, s is a **mixed strategy Nash equilibrium**
- When every s_i is deterministic, s is a **pure strategy Nash equilibrium**

Recap

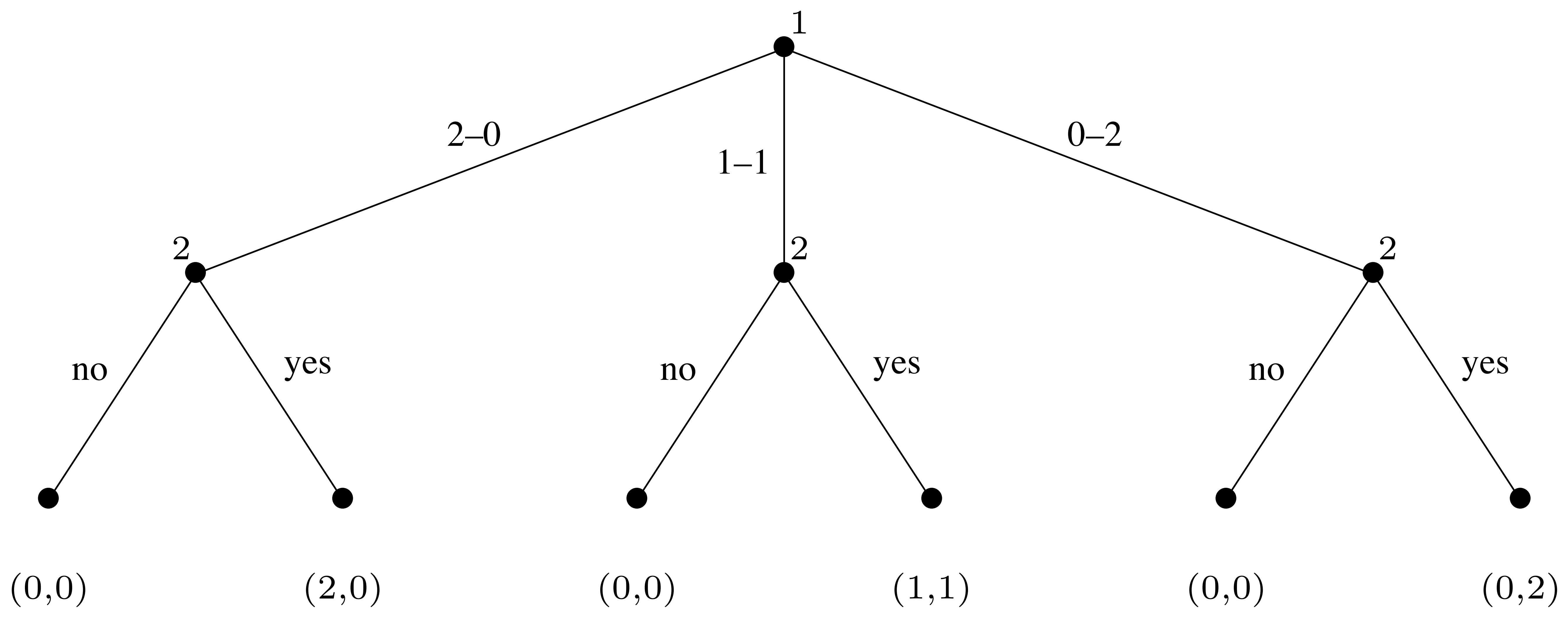
- **ϵ-Nash equilibria:** stable when agents have no deviation that gains them more than ε
- **Correlated equilibria:** stable when agents have **signals** from a possibly-correlated randomizing device
- **Linear programs** are a flexible encoding that can always be solved in **polytime**
- Finding a Nash equilibrium is **computationally hard** in general
- **Special cases** are efficiently computable:
    - Nash equilibria in zero-sum games
    - Maxmin strategies (and values) in two-player games
    - Correlated equilibrium

Lecture Outline

1. Recap
2. Extensive Form Games
3. Subgame Perfect Equilibrium
4. Backward Induction

Extensive Form Games

- Normal form games don't have any notion of **sequence**: all actions happen **simultaneously**
- The **extensive form** is a game representation that explicitly includes temporal structure (i.e., a **game tree**)



Perfect Information

There are two kinds of extensive form game:

1. **Perfect information:** Every agent **sees all actions** of the other players (including Nature)
   - e.g.: Chess, Checkers, Pandemic
   - This lecture!
2. **Imperfect information:** Some actions are **hidden**
   - Players may not know exactly where they are in the tree
   - e.g.: Poker, Rummy, Scrabble

Perfect Information Extensive Form Game

Definition:
A finite **perfect-information game in extensive form** is a tuple $G = (N, A, H, Z, \chi, \rho, \sigma, u)$, where

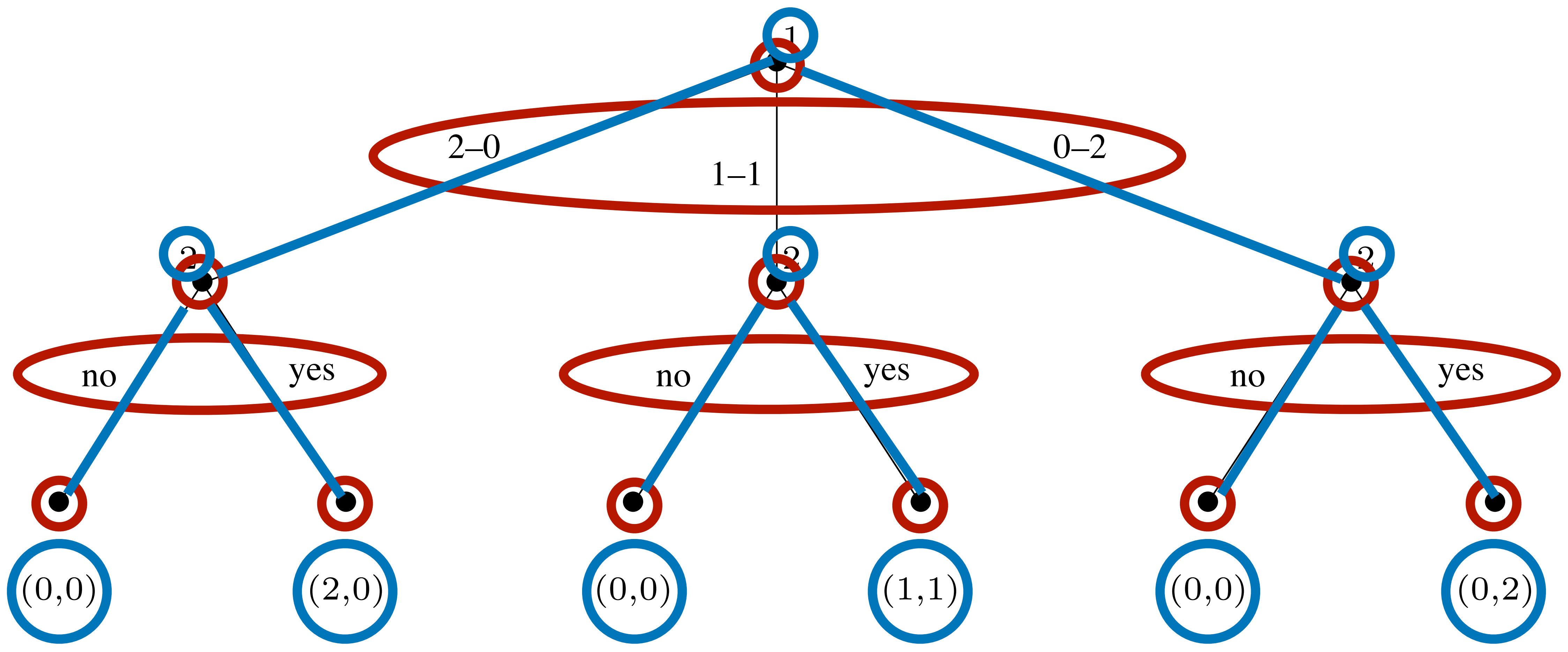
- N is a set of n **players**,
- A is a single set of **actions**,
- H is a set of nonterminal **choice nodes**,
- Z is a set of **terminal nodes** (disjoint from H),

$\chi : H \to 2^A$ is the **action function**,

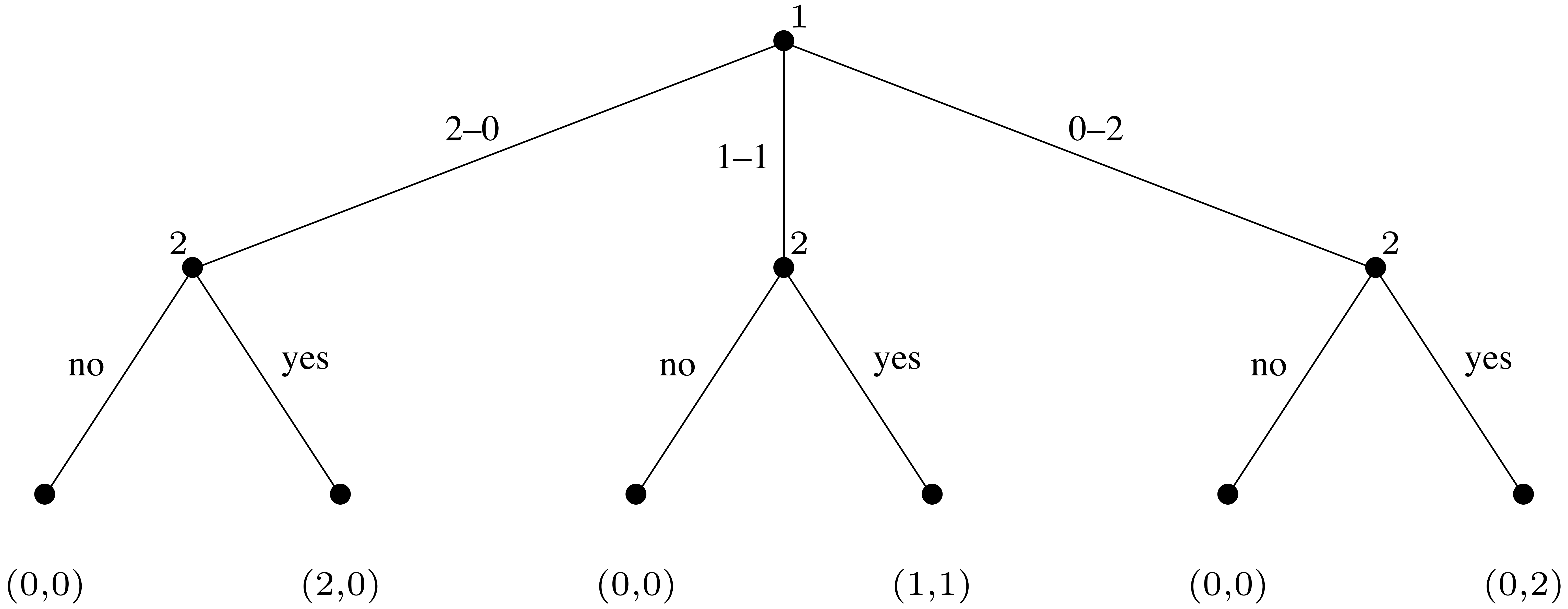
$\rho : H \to N$ is the **player function**,

$\sigma : H \times A \to H \cup Z$ is the **successor function**,

$u = (u_1, u_2, \ldots, u_n)$ is a profile of **utility functions** for each player, with $u_i : Z \to \mathbb{R}$.



Fun Game:
The Sharing Game



- Two siblings must decide how to share two $100 coins
- Sibling 1 suggests a division, then sibling 2 accepts or rejects
  - If rejected, nobody gets any coins.
- Play against 3 other people, once per person only

Pure Strategies

Question: What are the **pure strategies** in an extensive form game?

Definition:
Let $G = (N, A, H, Z, \chi, \rho, \sigma, u)$ be a perfect information game in extensive form. Then the **pure strategies of player** i consist of the cross product of actions available to player i at each of their choice nodes, i.e.,

$$\prod_{h \in H | \rho(h) = i} \chi(h).$$

Note: A pure strategy associates an action with **each** choice node, even those that will **never be reached.**

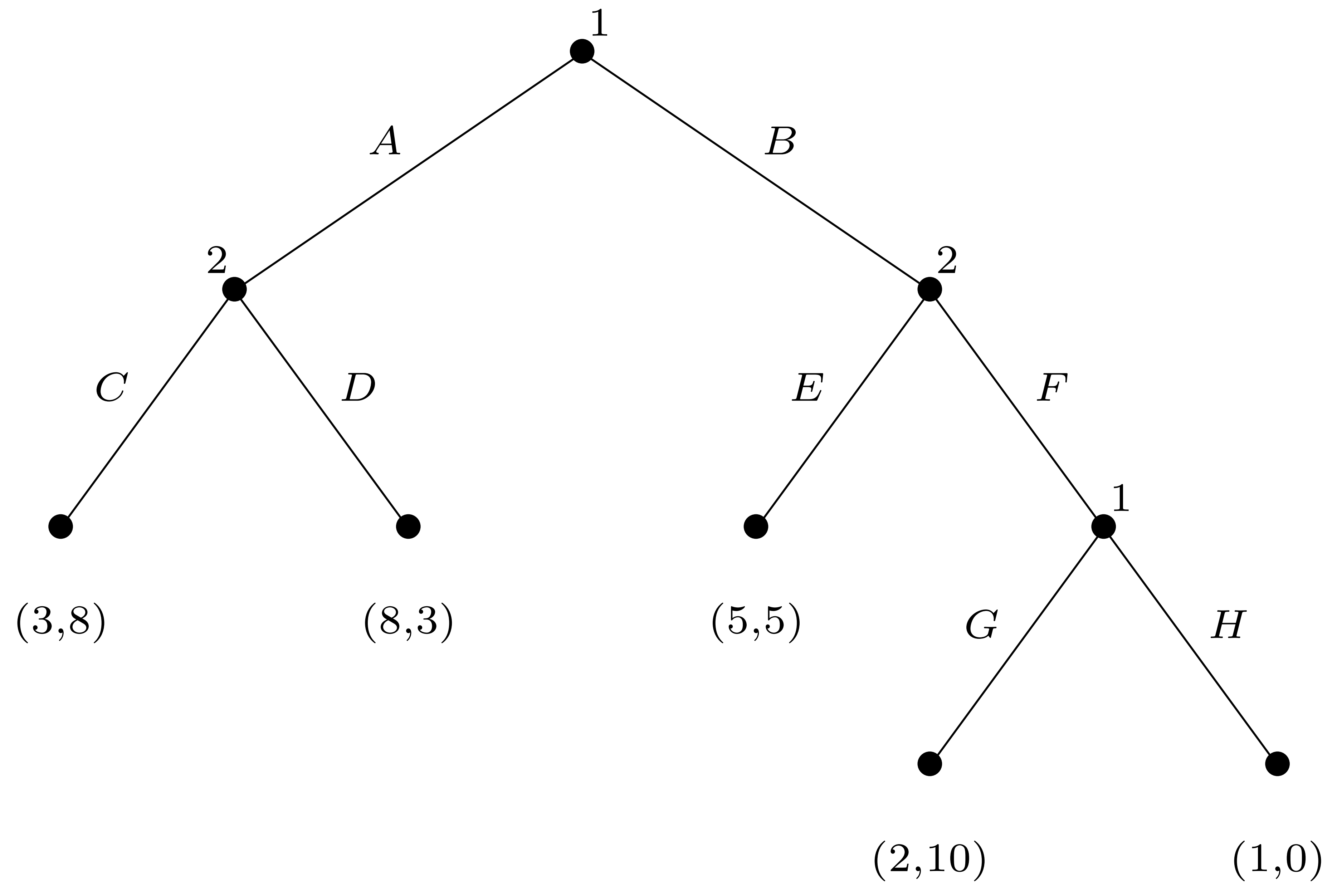
Pure Strategies Example

Question: What are the pure strategies for player 2?

- $\{(C, E), (C, F), (D, E), (D, F)\}$

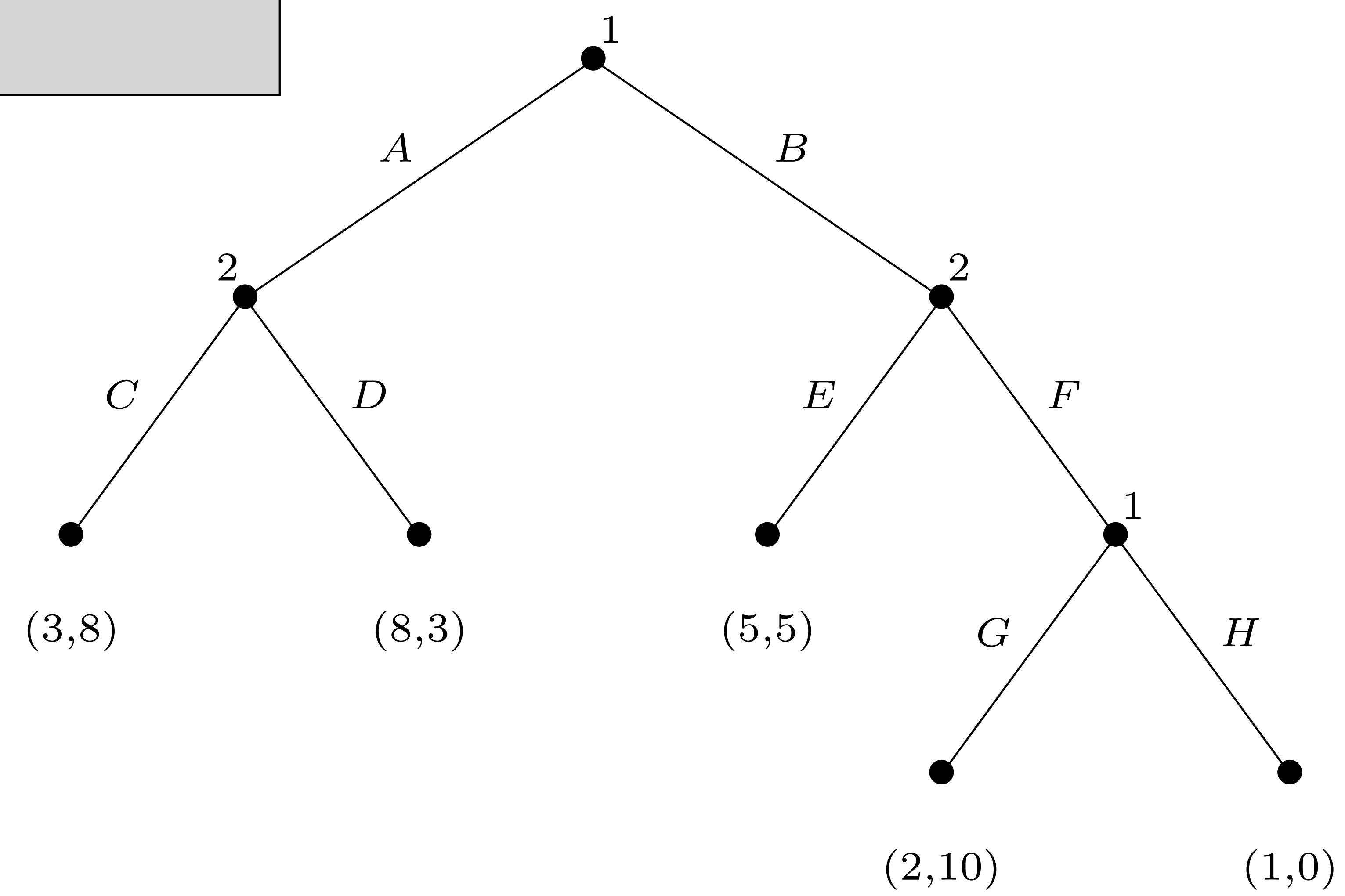
Question: What are the pure strategies for player 1?

- $\{(A, G), (A, H), (B, G), (B, H)\}$
- Note that these associate an action with the second choice node even when it can never be reached; e.g., (A, G) and (A, H).



Induced Normal Form

Question:
Which representation is more **compact**?



	C,E	C,F	D,E	D,F
A,G	3,8	3,8	8,3	8,3
A,H	3,8	3,8	8,3	8,3
B,G	5,5	2,10	5,5	2,10
B,H	5,5	1,0	5,5	1,0

- Any pair of pure strategies uniquely identifies a **terminal node**, which identifies a **utility** for each agent (**why?**)
- We have now defined a set of **agents**, **pure strategies**, and **utility functions**
- Any extensive form game defines a corresponding **induced normal form game**

Reusing Old Definitions

- We can plug our new definition of **pure strategy** into our existing definitions for:
    - Mixed strategy
    - Best response
    - Nash equilibrium (both pure and mixed strategy)

Question:

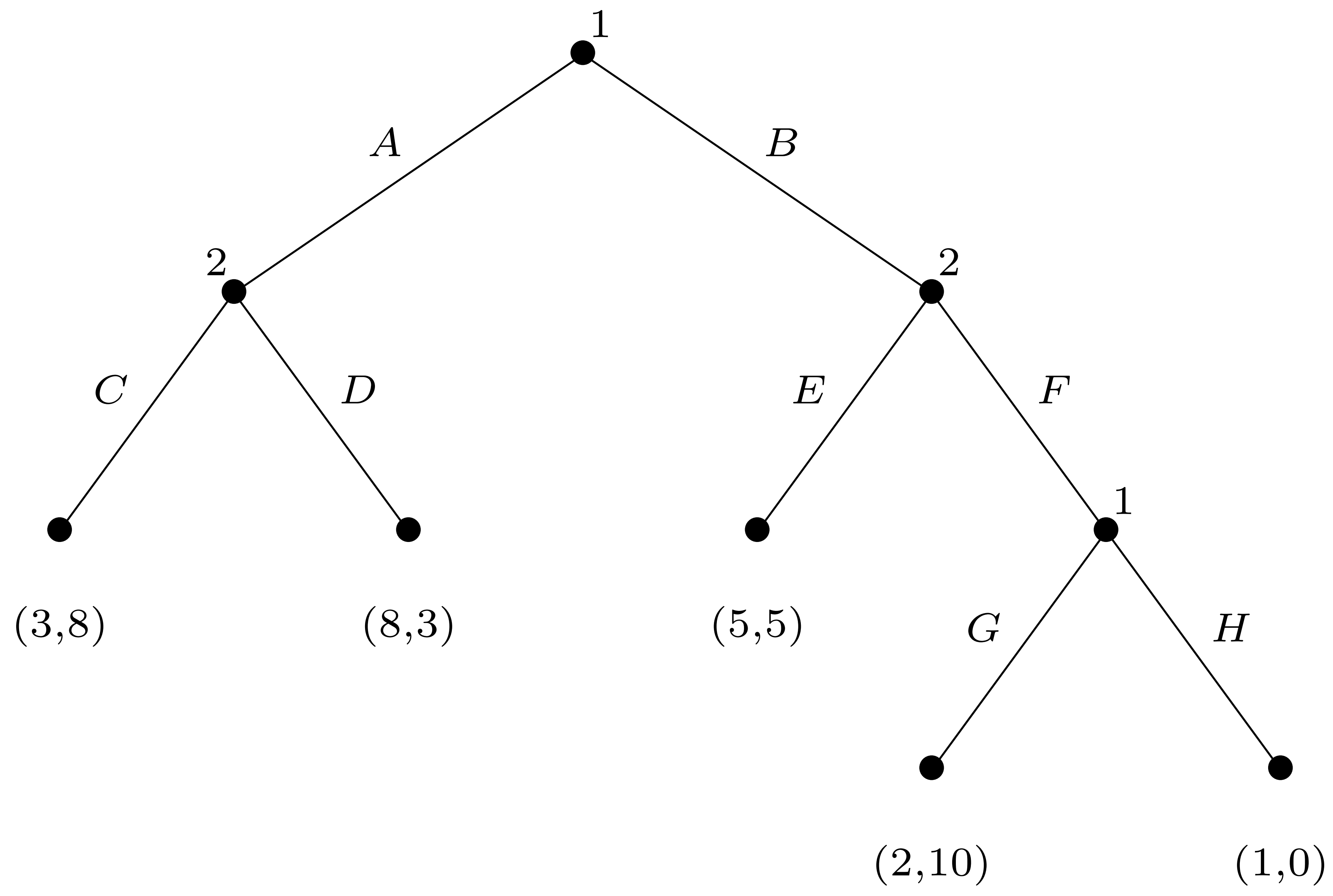
What is the definition of a **mixed strategy** in an extensive form game?

Pure Strategy Nash Equilibria

Theorem: [Zermelo 1913]
Every finite perfect-information game in extensive form has at least one **pure strategy Nash equilibrium**.

- Starting from the bottom of the tree, no agent needs to **randomize**, because they already know the best response
- There might be **multiple** pure strategy Nash equilibria in cases where an agent has multiple best responses at a **single choice node**

Pure Strategy Nash Equilibria



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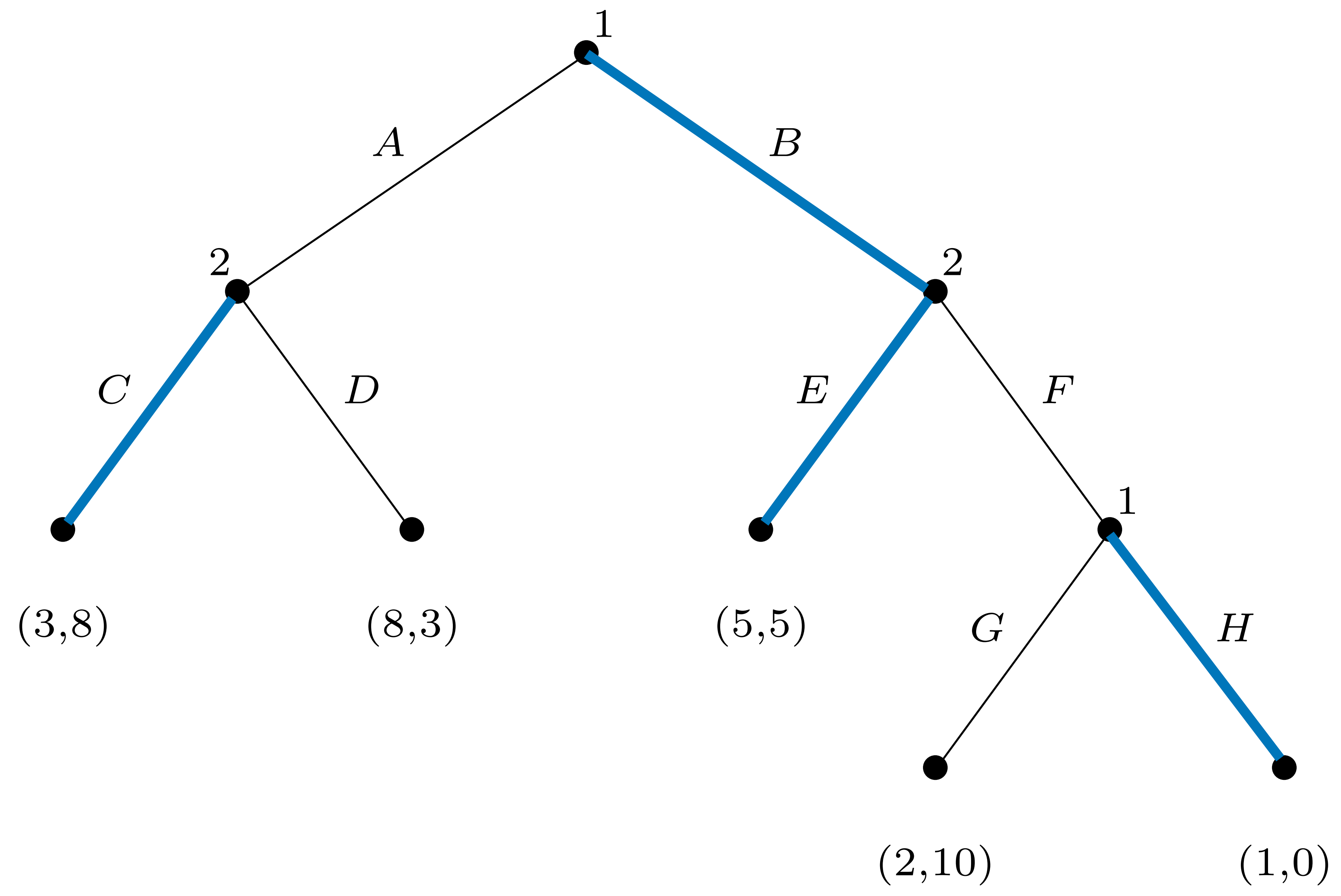
Question: What are the **pure-strategy Nash equilibria** of this game?

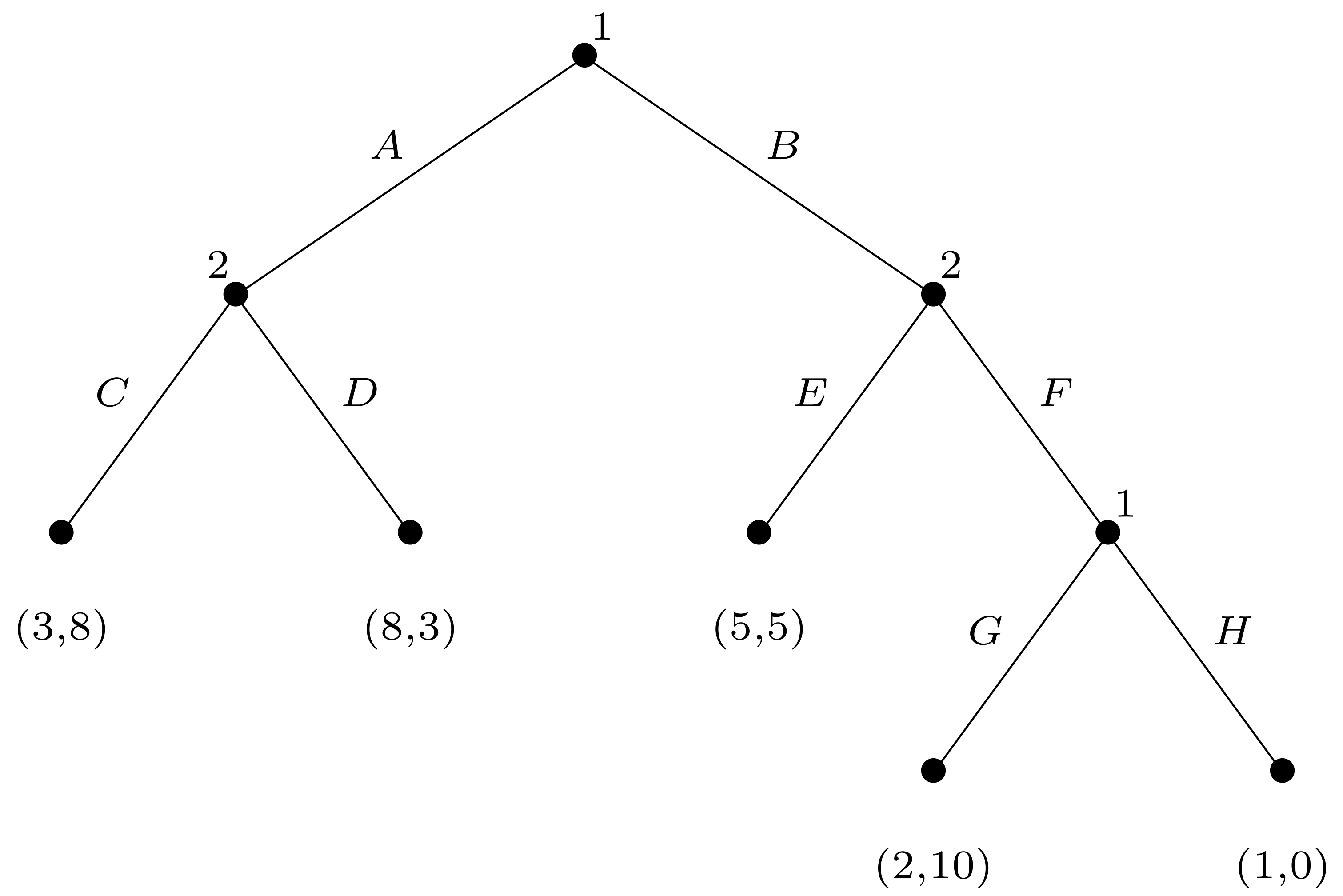
Question: Do any of them seem **implausible**?

Subgame Perfection, informally

- Some equilibria seem less **plausible** than others.
- (BH, CE): F has payoff 0 for player 2, because player 1 plays H, so their best response is to play E.
  - But why would player 1 play H if **they got to that choice node**?
  - The equilibrium relies on a threat from player 1 that is not **credible.**
- **Subgame perfect equilibria** are those that don't rely on non-credible threats.



Subgames

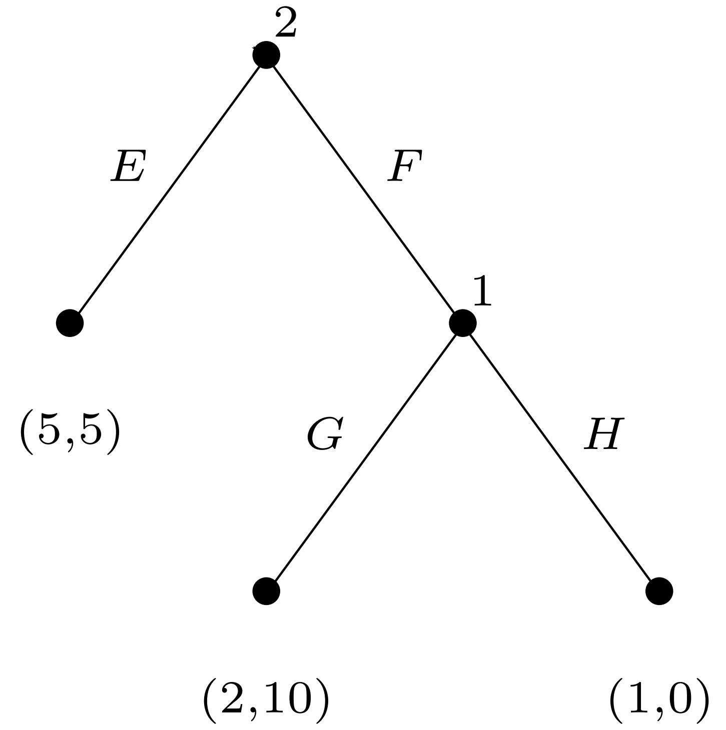


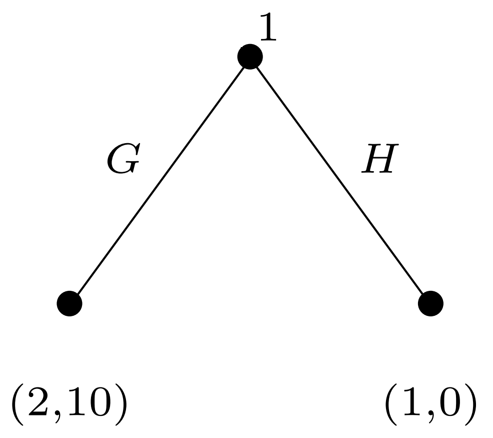
Definition:

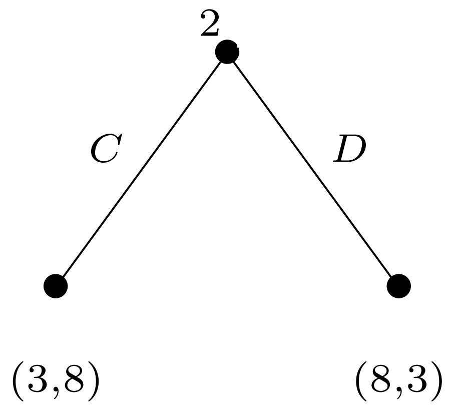
The **subgame of G rooted at h** is the restriction of G to the descendants of h.

Definition:

The **subgames of G** are the subgames of G rooted at h for every choice node $h \in H$.

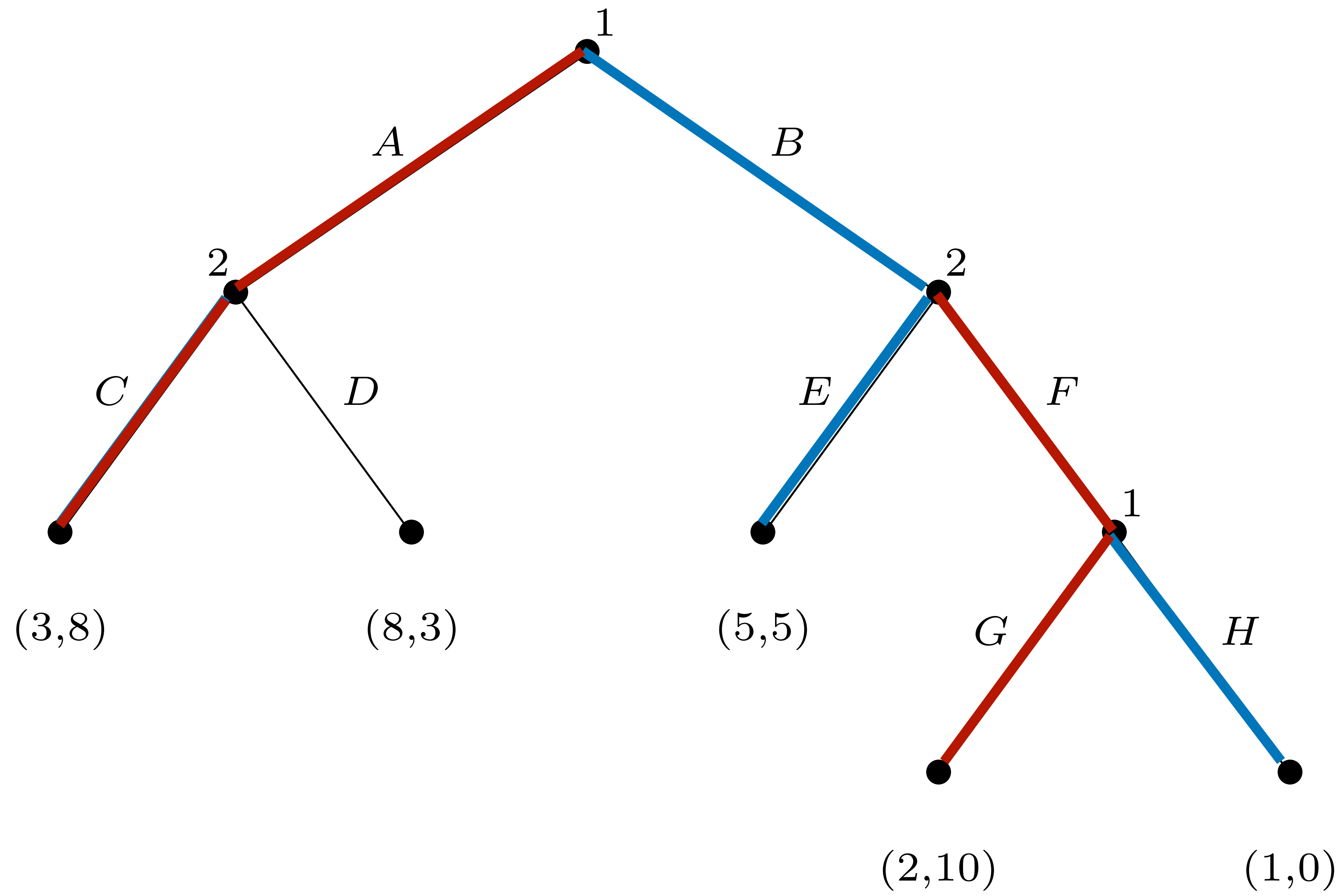






Subgame Perfect Equilibrium

Definition:
An strategy profile s is a **subgame perfect equilibrium** of G iff, for every subgame G' of G, the restriction of s to G' is a Nash equilibrium of G'.



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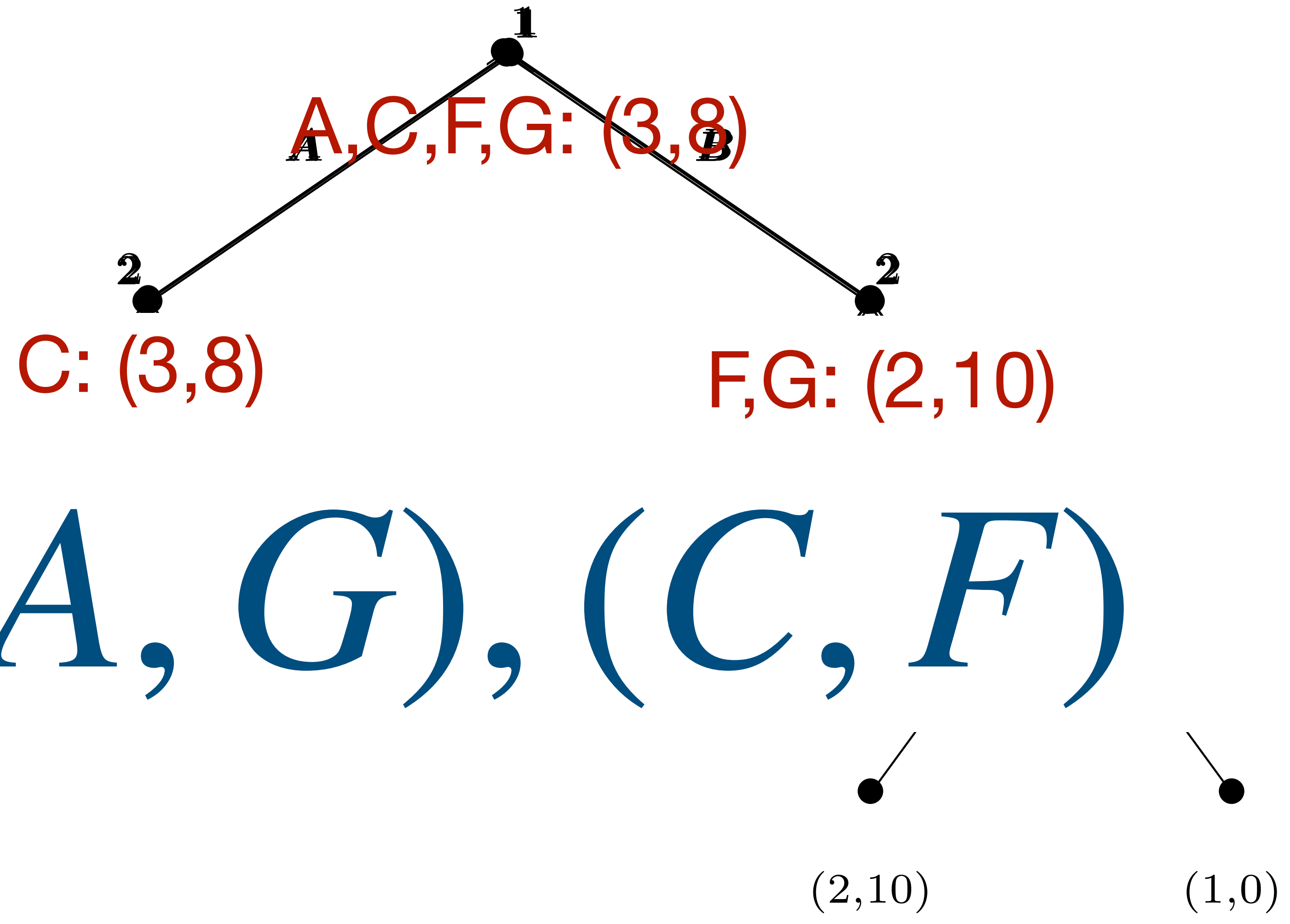
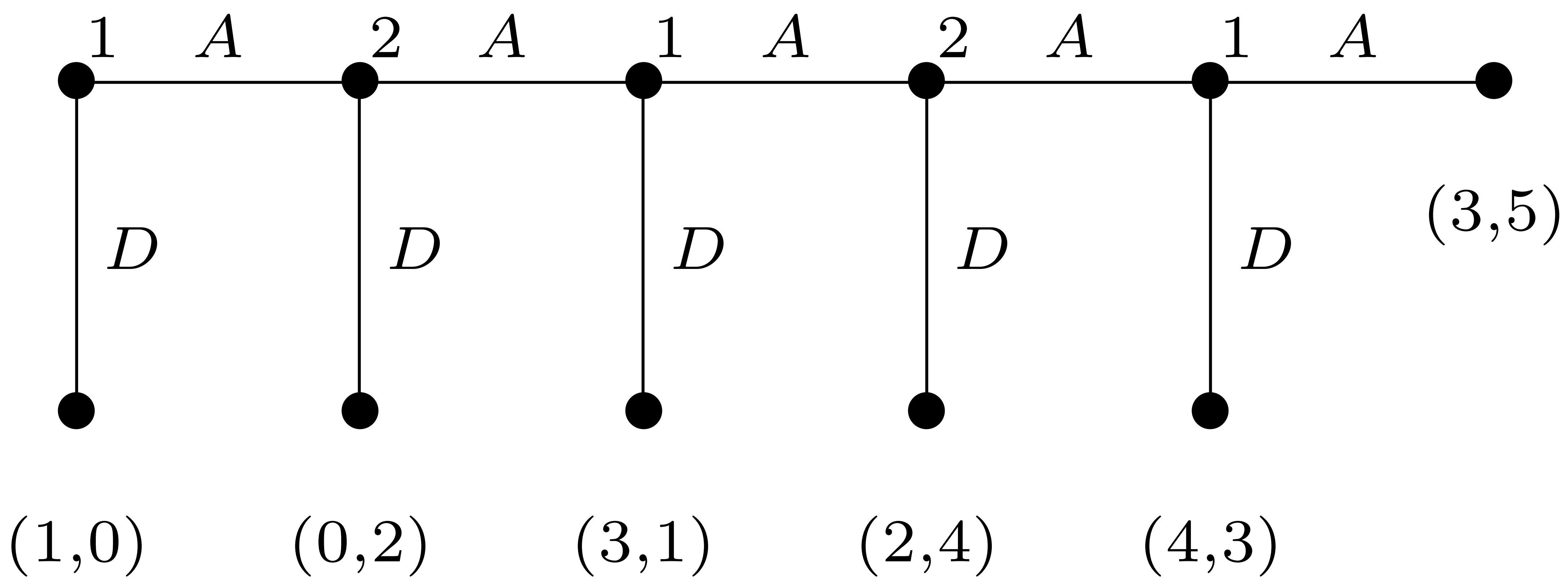
Backward Induction

- **Backward induction** is a straightforward algorithm that is guaranteed to compute a subgame perfect equilibrium.
- **Idea:** Replace subgames lower in the tree with their equilibrium values



$$(A, G), (C, F)$$

Fun Game: Centipede

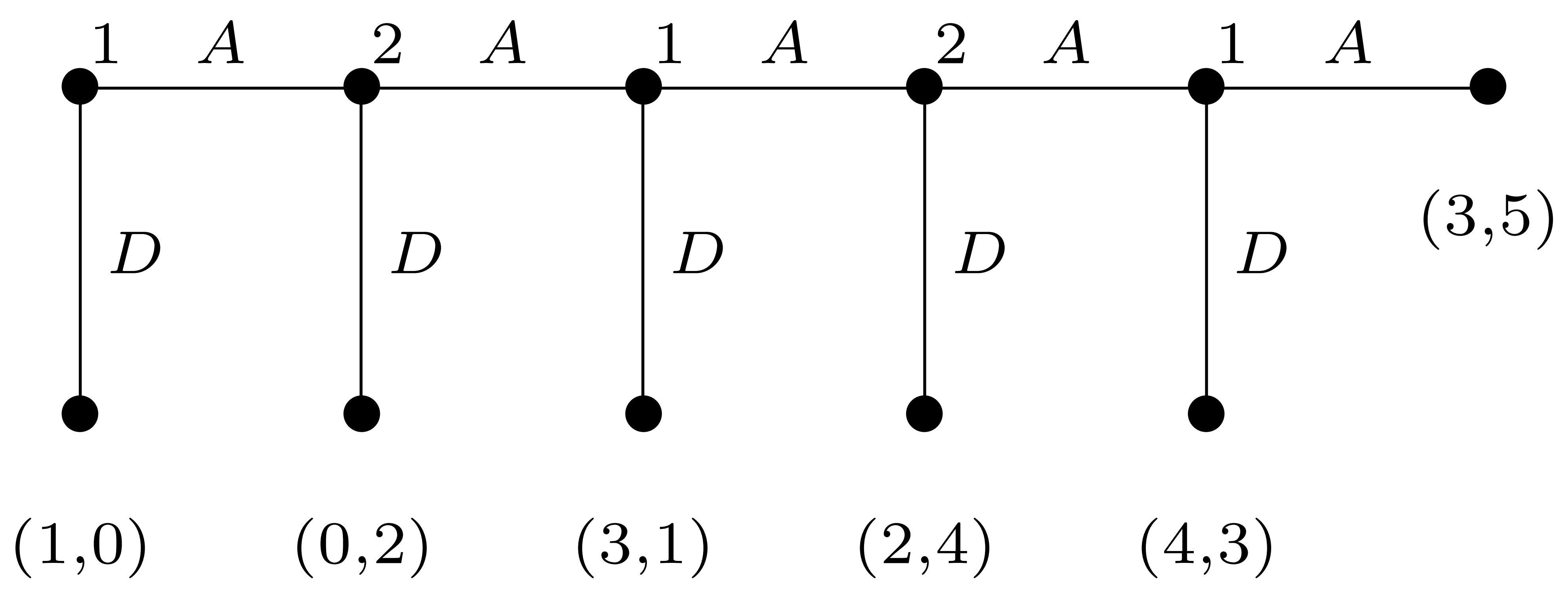


Question:

What is the unique **subgame perfect equilibrium** for Centipede?

- At each stage, one of the players can go **Across** or **Down.**
- If they go Down, the game ends.
- Play against three people! Try to play each role at least once.

Backward Induction Criticism



- The **unique** equilibrium is for each player to go Down **at the first opportunity.**
- **Empirically**, this is not how real people tend to play!
- **Theoretically**, what should you do if you arrive at an **off-path** node?
  - How do you update your beliefs to account for this probability 0 event?
  - If player 1 knows that you will update your beliefs in a way that causes you not to go down, then going down is no longer their only rational choice...

Summary

- **Extensive form games** allow us to represent sequential action
    - **Perfect information**: when we see everything that happens
- **Pure strategies** for extensive form games map **choice nodes** to **actions**
    - **Induced normal form** is the normal form game with these pure strategies
    - Notions of mixed strategy, best response, etc. **translate directly**
- **Subgame perfect equilibria** are those which do not rely on non-credible threats
    - Can always find a subgame perfect equilibrium using **backward induction**
    - But backward induction is theoretically and practically **complicated**