Game Theory Intro

CMPUT 654: Modelling Human Strategic Behaviour

S&LB §3.2-3.3.3

Recap: Utility Theory

- **Rational preferences** are those that satisfy axioms
- Representation theorems:
  - von Neumann & Morgenstern: Any rational preferences over **outcomes** can be represented by the maximization of the **expected value** of some **scalar** utility function
  - Savage: Any rational preferences over **acts** can be represented by maximization of the expected value of some scalar utility function with respect to **some probability distribution**

Lecture Outline

1. Recap
2. Noncooperative game theory
3. Normal form games
4. Solution concept: Pareto Optimality
5. Solution concept: Nash equilibrium
6. Mixed strategies

(Noncooperative) Game Theory

- **Utility theory** studies rational **single-agent** behaviour
- **Game theory** is the mathematical study of interaction between multiple **rational**, self-interested agents
  - **Self-interested:** Agents pursue only their **own preferences**
  - *Not* the same as "agents are psychopaths"! Their preferences may include the well-being of other agents.
  - Rather, the agents are **autonomous**: they decide on their own priorities independently.

Fun Game: Prisoner's Dilemma

	Cooperate	Defect
Cooperate	-1,-1	-5,0
Defect	0,-5	-3,-3

Two suspects are being questioned separately by the police.

- If they both remain silent (**cooperate** -- i.e., with each other), then they will both be sentenced to **1 year** on a lesser charge
- If they both implicate each other (**defect**), then they will both receive a reduced sentence of **3 years**
- If one defects and the other cooperates, the defector is given immunity (0 years) and the cooperator serves a full sentence of **5 years**.

Play the game with someone near you. Then find a new partner and play again. Play 3 times in total, against someone new each time.

Normal Form Games

The Prisoner's Dilemma is an example of a **normal form game**.
Agents make a single decision **simultaneously**, and then receive a payoff depending on the profile of actions.

Definition: Finite, n-person normal form game

- N is a set of n **players**, indexed by i
- $A = A_1 \times A_2 \times \ldots \times A_n$ is the set of **action profiles**
  - A_i is the **action set** for player i
- $u = (u_1, u2, \ldots, u_n)$ is a **utility function** for each player
  - $u_i : A \to \mathbb{R}$

Normal Form Games as a Matrix

	Cooperate	Defect
Cooperate	-1, -1, 1	-5, 0, 5
Defect	0,-5, 5	-3,-3, 3

Truthful

	Cooperate	Defect
Cooperate	-1,-1	-5,0
Defect	0,-5	-3,-3

	Cooperate	Defect
Cooperate	-1,-1, 1	-5, -5, 7
Defect	-5,-5, 7	-5, -5, 7

Lying

- Two-player normal form games can be written as a matrix with a tuple of utilities in each cell
- By convention, row player is first utility, column player is second
- Three-player normal form games can be written as a set of matrices, where the third player chooses the matrix

Games of Pure Competition (Zero-Sum Games)

Players have **exactly opposed** interests

- There must be precisely **two** players
  - Otherwise their interests can't be exactly opposed
- $u_1(a) + u_2(a) = c$ for all action profiles $a \in A$
  - $c = 0$ without loss of generality (**why?**)
- In a sense it's a **one-player game**
  - Only need to store a single number per cell
  - But also in a deeper sense, by the Minimax Theorem

Example: Matching Pennies

Row player wants to match, column player wants to mismatch

	Heads	Tails
Heads	1,-1	-1,1
Tails	-1,1	1,-1

Play against someone near you. Repeat 3 times.

Games of Pure Cooperation

Players have **exactly the same** interests.

- $u_i(a) = u_j(a)$ for all $i, j \in N$ and $a \in A$
- Can also write these games with one payoff per cell

Question: In what sense are these games **non-cooperative**?

Example Coordination Game

Which side of the road should you drive on?

	Left	Right
Left	1	-1
Right	-1	1

Play against someone near you.
Play 3 times in total, playing against someone new each time.

General Game:
Battle of the Sexes

The most interesting games are simultaneously both **cooperative and competitive**!

	Ballet	Soccer
Ballet	2, 1	0, 0
Soccer	0, 0	1, 2

Play against someone near you.
Play 3 times in total, playing against someone new each time.

Optimal Decisions in Games

- In single-agent decision theory, the key notion is
  optimal decision: a decision that maximizes the agent's expected utility
- In a multiagent setting, the notion of optimal strategy is **incoherent**
  - The best strategy **depends** on the strategies of others

Solution Concepts

- From the viewpoint of an **outside observer**, can some outcomes of a game be labelled as **better** than others?
  - We have no way of saying one agent's interests are more important than another's
  - We can't even **compare** the agents' utilities to each other, because of affine invariance! We don't know what "**units**" the payoffs are being expressed in.
- Game theorists identify certain subsets of outcomes that are interesting in one sense or another. These are called **solution concepts**.

Pareto Optimality

- Sometimes, some outcome o is **at least as good** for any agent as outcome o', and there is some agent who **strictly prefers** o to o'.
  - *Example:* $o' =$ "Everyone gets pie", vs.
    $o =$ "Everyone gets pie *and also* Alice gets cake"
  - In this case, o seems defensibly better than o'

Questions:

1. Can a game have **more than one** Pareto-optimal outcome?
2. Does every game have **at least one** Pareto-optimal outcome?

Definition: o **Pareto dominates** o' when
$o \succeq_i o'$ for **all** $i \in N$ and $o \succ_i o'$ for **some** $i \in N$.

Definition:
An outcome o^* is **Pareto optimal** if no other outcome Pareto dominates it.

Pareto Optimality Examples

Definition: o **Pareto dominates** o' when
$o \succeq_i o'$ for **all** $i \in N$ and $o \succ_i o'$ for **some** $i \in N$.

Definition:
An outcome o^* is **Pareto optimal** if no other outcome Pareto dominates it.

$\begin{bmatrix} 9 \\ 8 \\ 7 \end{bmatrix}$ Pareto-dominates $\begin{bmatrix} 8 \\ 2 \\ 7 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$ Pareto-dominates $\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$

Does $\begin{bmatrix} 9 \\ 8 \\ 7 \end{bmatrix}$ Pareto-dominate $\begin{bmatrix} 1 \\ 1 \\ 8 \end{bmatrix}$?

Out of $\left\{ \begin{bmatrix} 9 \\ 8 \\ 7 \end{bmatrix}, \cancel{\begin{bmatrix} 8 \\ 2 \\ 7 \end{bmatrix}}, \cancel{\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}}, \cancel{\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}}, \begin{bmatrix} 1 \\ 1 \\ 8 \end{bmatrix} \right\}$ which outcomes are Pareto-optimal?

Best Response

- Which **actions** are better from an **individual agent's** viewpoint?
- That depends on what the other agents are doing!

Notation:

$$a_{-i} \doteq (a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$$
$$a = (a_i, a_{-i})$$

Definition:

$$BR_i(a_{-i}) \doteq \{a_i^* \in A_i \mid u_i(a^*, a_{-i}) \geq u_i(a_i, a_{-i}) \;\; \forall a_i \in A_i\}$$

is the set of agent i's **pure best responses** to a_{-i}.

Nash Equilibrium

- Best response is not, in itself, a solution concept
  - In general, agents won't know what the other agents will do
  - But we can use it to define a solution concept
- A Nash equilibrium is a **stable** outcome: one where no agent regrets their actions

Definition:
An action profile $a \in A$ is a (pure strategy) **Nash equilibrium** iff

$$\forall i \in N: \quad a_i \in BR_{-i}(a_{-i})$$

Questions:

1. Can a game have **more than one** pure strategy Nash equilibrium?
2. Does every game have **at least one** pure strategy Nash equilibrium?

Nash Equilibria of Examples

The only equilibrium of Prisoner's Dilemma is also the *only* outcome that is Pareto-dominated!

	Coop.	Defect
Coop.	-1,-1	-5,0
Defect	0,-5	-3,-3

	Left	Right
Left	1	-1
Right	-1	1

	Ballet	Soccer
Ballet	2, 1	0, 0
Soccer	0, 0	1, 2

	Heads	Tails
Heads	1,-1	-1,1
Tails	-1,1	1,-1

Mixed Strategies

- So far, we have been assuming that agents play a single action **deterministically**
  - But that's a pretty bad idea in, e.g., Matching Pennies

Definition:

- A **strategy** s_i for agent i is any probability distribution over the set A_i, where each action a_i is played with probability $s_i(a_i)$.
  - **Pure strategy**: only a single action is played
  - **Mixed strategy**: randomize over multiple actions
- Set of i's strategies: $S_i \doteq \Delta(A_i)$
- Set of **strategy profiles**: $S \doteq S_1 \times \ldots \times S_n$

Utility Under Mixed Strategies

The utility under a mixed strategy profile is **expected utility** (**why?**)

- Because we assume agents are decision-theoretically rational
- We assume that the agents randomize **independently**

Definition:

For any mixed strategy profile s,

$$u_i(s) = \sum_{a \in A} \Pr(a \mid s) u_i(a),$$

where $\Pr(a \mid s) = \prod_{j \in N} s_j(a_j).$

Best Response and Nash Equilibrium

Definition:
The set of i's **best responses** to a strategy profile $s_{-i} \in S_{-i}$ is

$$BR_i(s_{-i}) \doteq \{s_i^* \in S \mid u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i}) \;\; \forall s_i \in S_i\}$$

Definition:
A strategy profile $s \in S$ is a **Nash equilibrium** iff

$$\forall i \in N : \quad s_i \in BR_{-i}(s_{-i})$$

- When at least one s_i is mixed, s is a **mixed strategy Nash equilibrium**
- When every s_i is deterministic, s is a **pure strategy Nash equilibrium**

Nash's Theorem

Theorem: [Nash 1951]
Every game with a finite number of players and action profiles has at least one Nash equilibrium.

Proof idea:

1. Brouwer's fixed-point theorem guarantees that any continuous function from a simpletope to itself has a fixed point.
2. Construct a continuous function $f : S \to S$ whose fixed points are all Nash equilibria.

- NB: A **simpletope** is a product of simplices, so S is a simpletope

Interpreting
Mixed Strategy Nash Equilibrium

What does it even mean to say that agents are playing a mixed strategy Nash equilibrium?

- They truly are **sampling a distribution** in their heads, perhaps to **confuse** their opponents (e.g., soccer, other zero-sum games)
- The distribution represents the **other agents' uncertainty** about what the agent will do
- The distribution is the **empirical frequency** of actions in repeated play
- The distribution is the frequency of a pure strategy in a **population** of pure strategies (i.e., every individual plays a pure strategy)

Summary

- Game theory studies the **interactions of rational agents**
    - Canonical representation is the **normal form game**
- Game theory uses **solution concepts** rather than optimal behaviour
    - "Optimal behaviour" is not clear-cut in multiagent settings
    - **Pareto optimal:** no agent can be made better off without making some other agent worse off
    - **Nash equilibrium:** no agent regrets their strategy given the choice of the other agents' strategies