# Utility Theory

CMPUT 654: Modelling Human Strategic Behaviour

S&LB §3.1

# Recap: Course Essentials

## jrwright.info/bgtcourse/

- This is the **main source** for information about the class
- Slides, readings, deadlines

## eClass

- This is where **assignments** are posted and handed in
- There is also a **class forum** for questions and discussions about course material

# Utility, informally

A utility function is a real-valued function that indicates how much an agent **prefers** an outcome.

**Rational agents act to maximize their expected utility.**

**Nontrivial** claim:

1. Why should we believe that an agent's preferences can be adequately represented by a **single number**?
2. Why should agents maximize **expected value** rather than some other criterion?

Von-Neumann and Morgenstern's Theorem shows when these are true.

# Outline

1. Informal statement
2. Theorem statement (von Neumann & Morgenstern)
3. Proof sketch
4. Fun game!
5. Representation theorem (Savage)

# Formal Setting: Outcome

**Definition:** Let $O$ be a set of **outcomes**:

$$O = Z \cup \Delta(O)$$

Not a typo!

where $Z$ is some set of "actual outcomes", and

$\Delta(X)$ represents the set of **lotteries** over **finite** subsets of $X$:

$$[p_1 : x_1, \ldots, p_k : x_k]$$

with $\sum_{j=1}^{k} p_j = 1$ and $x_j \in X \;\; \forall 1 \leq j \leq k$

# Formal Setting: Preference Relation

A preference relation is a relationship between outcomes.

**Definition**

For a specific **preference relation** $\succeq$, write:

1. $o_1 \succeq o_2$ if the agent **weakly prefers** $o_1$ to $o_2$,
2. $o_1 \succ o_2$ if the agent **strictly prefers** $o_1$ to $o_2$,
3. $o_1 \sim o_2$ if the agent is **indifferent** between $o_1$ and $o_2$.

# Formal Setting

**Definition**
A **utility function** is a function $u : O \to \mathbb{R}$. A utility function **represents** a preference relation $\succeq$ iff:

1. $o_1 \succeq o_2 \iff u(o_1) \geq u(o_2)$, and

2. $u([p_1 : o_1, \ldots, p_k : o_k]) = \sum_{j=1}^{k} p_j u(o_j)$.

# Representation Theorem

**Theorem:** [von Neumann & Morgenstern, 1944]
Suppose that a preference relation $\succeq$ satisfies the axioms **Completeness**, **Transitivity**, **Monotonicity**, **Substitutability**, **Decomposability**, and **Continuity**.

Then there exists a function $u : O \to \mathbb{R}$ such that

1. $o_1 \succeq o_2 \iff u(o_1) \geq u(o_2)$, and
2. $u([p_1 : o_1, \ldots, p_k : o_k]) = \sum_{j=1}^{k} p_j u(o_j)$.

That is, there exists a utility function that **represents** $\succeq$.

# Completeness and Transitivity

**Definition (Completeness):**

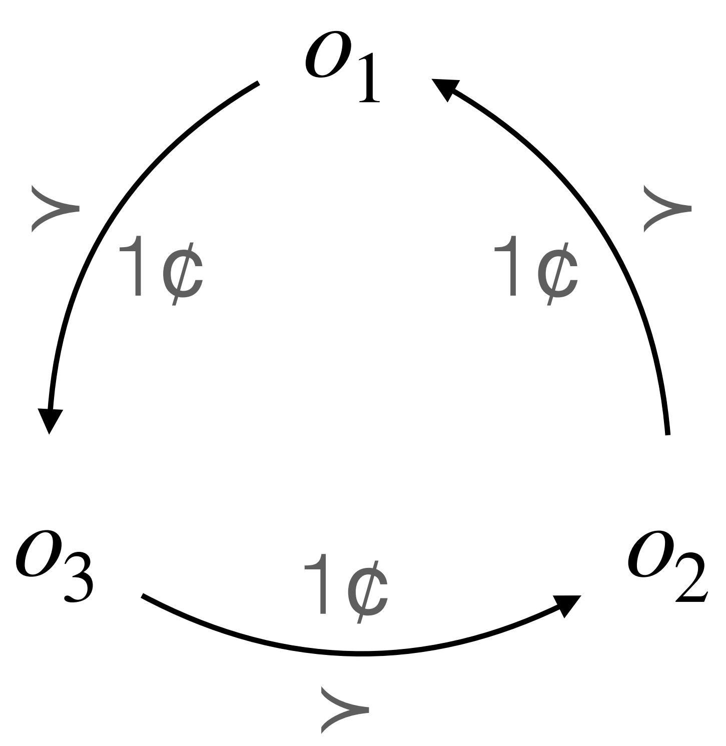
$$\forall o_1, o_2 : (o_1 \succ o_2) \vee (o_1 \prec o_2) \vee (o_1 \sim o_2)$$

**Definition (Transitivity):**

$$\forall o_1, o_2 : (o_1 \succeq o_2) \wedge (o_2 \succeq o_3) \implies o_1 \succeq o_3$$

**Question:** Should we buy these axioms?

# Transitivity Justification: Money Pump




- Suppose that $(o_1 \succ o_2)$ and $(o_2 \succ o_3)$ and $(o_3 \succ o_1)$.
- Starting from $o_3$, you are willing to pay 1¢ (say) to switch to $o_2$
- But from $o_2$, you should be willing to pay 1¢ to switch to $o_1$
- But from $o_1$, you should be willing to pay 1¢ to switch back to $o_3$ again...

# Monotonicity

**Definition (Monotonicity):**
If $o_1 \succ o_2$ and $p > q$, then

$$[p : o_1, (1-p) : o_2] \succ [q : o_1, (1-q) : o_2].$$

You should prefer a 90% chance of getting $1000 to
a 50% chance of getting $1000.

**Questions:**

1. Does this axiom depend on the agent's risk attitudes?
2. Must this be true of all rational preferences?

# Substitutability

**Definition (Substitutability):**

If $o_1 \sim o_2$, then for all sequences $o_3, \ldots, o_k$ and $p, p_3, \ldots, p_k$ with

$$p + \sum_{j=3}^{k} p_j = 1,$$

$$[p : \boxed{o_1}, p_3 : o_3, \ldots, p_k : o_k] \sim [p : \boxed{o_2}, p_3 : o_3, \ldots, p_k : o_k]$$

If I like apples and bananas equally, then I should be indifferent between a 30% chance of getting an apple and a 30% chance of getting a banana.

**Question:** Should we buy this axiom?

# Decomposability aka "No Fun in Gambling"

**Definition (Decomposability)**:
Let $P_\ell(o)$ denote the probability that lottery $\ell$ selects outcome $o$.

If $P_{\ell_1}(o_j) = P_{\ell_2}(o_j)\ \ \forall o_j \in O$, then $\ell_1 \sim \ell_2$.

**Example:**

Let $\ell_1 = [0.5 : [0.5 : o_1,\ 0.5 : o_2],\ 0.5 : o_3]$
Let $\ell_2 = [0.25 : o_1,\ 0.25 : o_2,\ 0.5 : o_3]$

Then $\ell_1 \sim \ell_2$, because

$$
\begin{aligned}
P_{\ell_1}(o_1) &= 0.5 \times 0.5 = 0.25 && = P_{\ell_2}(o_1) \\
P_{\ell_1}(o_2) &= 0.5 \times 0.5 = 0.25 && = P_{\ell_2}(o_2) \\
P_{\ell_1}(o_3) &= 0.5 && = P_{\ell_2}(o_3)
\end{aligned}
$$

# Continuity

**Definition (Continuity):**

If $o_1 \succ o_2 \succ o_3$, then $\exists p \in [0,1]$ such that

$$o_2 \sim [p : o_1, (1-p) : o_3]$$

# Proof Sketch: Construct the utility function

1. If $\succeq$ satisfies Completeness, Transitivity, Monotonicity, Decomposability, then for every $o_1 \succ o_2 \succ o_3$, there exists some $p$ such that:
   (a) $o_2 \succ [q : o_1, (1-q) : o_3] \;\; \forall q < p$, and
   (b) $o_2 \prec [q : o_1, (1-q) : o_3] \;\; \forall q > p$.
2. If $\succeq$ additionally satisfies Continuity, then
   $$\exists p : o_2 \sim [p : o_1, (1-p) : o_3].$$
3. Choose **maximal** $o^+ \in O$ and **minimal** $o^- \in O$.
4. Construct $u(o) = p$ such that $o \sim [p : o^+, (1-p) : o^-]$.

**Question:** Are $o^+$ and $o^-$ guaranteed to exist?

# Proof sketch:
# Check the properties

1. $o_1 \succeq o_2 \iff u(o_1) \geq u(o_2)$

$u(o) = p$ such that $o \sim [p : o^+, (1-p) : o^-]$.

# Proof sketch:
# Check the properties

2. $u([p_1 : o_1, \ldots, p_k : o_k]) = \sum_{j=1}^{k} p_j u(o_j)$

(i) Let $u^* = u([p_1 : o_1, \ldots, p_k : o_k])$

(ii) Replace $o_j$ with $\ell_j = [u(o_j) : o^+, (1 - u(o_j)) : o^-]$, giving
$[p_1 : \ell_1, \ldots, p_k : \ell_k] = [p_1 : [u(o_1) : o^+, (1 - u(o_1)) : o^-], \ldots, p_k : [u(o_k) : o^+, (1 - u(o_k)) : o^-]]$

(iii) **Question:** What is $u([p_1 : \ell_1, \ldots, p_k : \ell_k])$?

$u([p_1 : \ell_1, \ldots, p_k : \ell_k]) = u^*$

(iv) **Question:** What is the probability of getting $o^+$ in $[p_1 : \ell_1, \ldots, p_k : \ell_k]$?

$\sum_{j=1}^{k} \left(p_j \times u(o_j)\right)$

(v) Construct $\ell^* = \left[\sum_{j=1}^{k} \left(p_j \times u(o_j)\right) : o^+, \left(1 - \sum_{j=1}^{k} \left(p_i \times u(o_j)\right)\right) : o^-\right]$

$u(\ell^*) = \sum_{j=1}^{k} \left(p_j \times u(o_j)\right)$

(vi) Observe that $[p_1 : \ell_1, \ldots, p_k : \ell_k] \sim \ell^*$ (**why?**)

$u([p_1 : \ell_1, \ldots, p_k : \ell_k]) = u^* = u(\ell^*) = \sum_{j=1}^{k} \left(p_j \times u(o_j)\right)$ ∎

# Caveats & Details

Utility functions are **not uniquely defined.** (**Why?**)

- Invariant to affine transformations (i.e., $m > 0$):

$$\mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)] \iff X \succeq Y$$

$$\iff \mathbb{E}[mu(X) + b] \geq \mathbb{E}[mu(Y) + b] \iff X \succeq Y$$

This means we're not stuck with a range of [0,1]!

# Caveats & Details

The proof depended on **minimal** and **maximal** elements of $O$, but that is not critical.

Construction for **unbounded** outcomes/preferences:

1. Pick two outcomes $o_s \prec o_e$. Construct utility for all outcomes $o_s \preceq o \preceq o_e$:

$$u : \{o \in O \mid o_s \preceq o \preceq o_e\} \rightarrow [0,1]$$

2. For outcomes $o'$ outside that range, choose $o_{s'} \prec o' \prec o_s \prec o_e \prec o_{e'}$.
3. Construct utility $u' : \{o \in O \mid o_{s'} \preceq o \preceq o_{e'}\} \rightarrow [0,1]$.
4. Find $m > 0$ and $b \in \mathbb{R}$ such that $mu'(o_s) + b = u(o_s)$ and $mu'(o_e) + b = u(o_e)$.
5. Let $u(o) = mu'(o) + b$ for all $o \in \{o' \in O \mid o_{s'} \preceq o' \preceq o_{e'}\}$.

# Fun game:
# Buying lottery tickets

Write down the following numbers:

1. How much would you pay for the lottery
   [0.3 : \$5, 0.3 : \$7, 0.4 : \$9]?
2. How much would you pay for the lottery
   [$p$ : \$5, $q$ : \$7, (1 - $p$ - $q$) : \$9]?
3. How much would you pay for the lottery
   [$p$ : \$5, $q$ : \$7, (1 - $p$ - $q$) : \$9]
   *if you knew the last seven draws had been 5,5,7,5,9,9,5?*

# Beyond von Neumann & Morgenstern

- The first step of the fun game was a good match to the utility theory we just learned.
  - **Question:** If two agents have different prices for [0.3 : \$5,  0.3 : \$7,  0.4 : \$9], what does that say about their utility functions for money?
- The second and third steps, not so much!
  - **Question:** If two agents have different prices for [$p$ : \$5,  $q$ : \$7,  (1 - $p$ - $q$) : \$9], what does that say about their **utility functions**?
  - What if two people have the same prices for step 2 but different prices once they hear what the last few draws were?

# Another Formal Setting

- **States**: Set $S$ of elements $s, s', \ldots$ with subsets $A, B, C, \ldots$
- **Consequences**: Set $F$ of elements $f, g, h, \ldots$
- **Acts**: Arbitrary functions $\mathbf{f} : S \to F$
- Preference relation $\succeq$ **between acts**
- $$(\mathbf{f} \succeq \mathbf{g} \text{ given } B) \iff$$
  $\mathbf{f}' \succeq \mathbf{g}'$ for every $\mathbf{f}', \mathbf{g}'$ that agree with $\mathbf{f}, \mathbf{g}$ respectively on $B$ and each other on $\overline{B}$

# Another Representation Theorem

**Theorem:** [Savage, 1954]

Suppose that a preference relation $\succeq$ satisfies postulates P1-P6.

Then there exists a utility function $U$ $\boxed{\text{and a probability measure } P}$ such that

$$\mathbf{f} \succeq \mathbf{g} \iff \sum_i P[B_i]U[f_i] \geq \sum_i P[B_i]U[g_i].$$

# Postulates

**P1** $\succeq$ is a simple order

**P2** $\forall \mathbf{f}, \mathbf{g}, B : (\mathbf{f} \succeq \mathbf{g} \text{ given } B) \vee (\mathbf{g} \succeq \mathbf{f} \text{ given } B)$

**P3** $(\mathbf{f}(s) = g \wedge \mathbf{f}'(s) = g' \;\; \forall s \in B) \implies (\mathbf{f} \succeq \mathbf{f}' \text{ given } B \iff g \succeq g')$

**P4** For every $A, B$, either $A \leq B$ or $B \leq A$ (see D4)

**P5** It is false that for every $f, f'$, $f \succeq f'$.

**P6** For all $\mathbf{g} \succ \mathbf{h}$ and consequence $f$, there exists a partition of $S$ such that the consequence of either $\mathbf{g}$ or $\mathbf{h}$ can be replaced by $f$ without changing the ordering of the two acts.

# Summary

- Using very simple axioms about **preferences over lotteries**, utility theory proves that rational agents ought to act **as if** they were maximizing the **expected value** of a real-valued function.
  - **Rational** agents are those whose behaviour satisfies a certain set of **axioms**
  - *If you don't buy the axioms, then you shouldn't buy that this theorem is about rational behaviour*
- Can extend beyond this to “subjective” probabilities, using axioms about **preferences over uncertain "acts"** that do not describe how agents manipulate probabilities.