Imperfect Information Extensive Form Games

CMPUT 654: Modelling Human Strategic Behaviour

S&LB §5.2-5.2.2

Assignment Hint:
Mixed Strategy Nash by Hand

- Recall that if we know the **support** of an equilibrium in a two-player game we can compute its equilibrium with an LP

- For small games, you can just solve a system of equations for the probabilities of each action by hand.

$$\sum_{a_{-i} \in \sigma_{-i}} s_{-i}(a_{-i}) u_i(a_i, a_{-i}) = v_i \qquad \forall i \in \{1,2\}, a_i \in \sigma_i$$

$$\sum_{a_{-i} \in \sigma_{-i}} s_{-i}(a_{-i}) u_i(a_i, a_{-i}) \leq v_i \qquad \forall i \in \{1,2\}, a_i \notin \sigma_i$$

$$s_i(a_i) \geq 0 \qquad \forall i \in \{1,2\}, a_i \in \sigma_i$$

$$s_i(a_i) = 0 \qquad \forall i \in \{1,2\}, a_i \notin \sigma_i$$

$$\sum_{a_i \in A_i} s_i(a_i) = 1 \qquad \forall i \in \{1,2\}$$

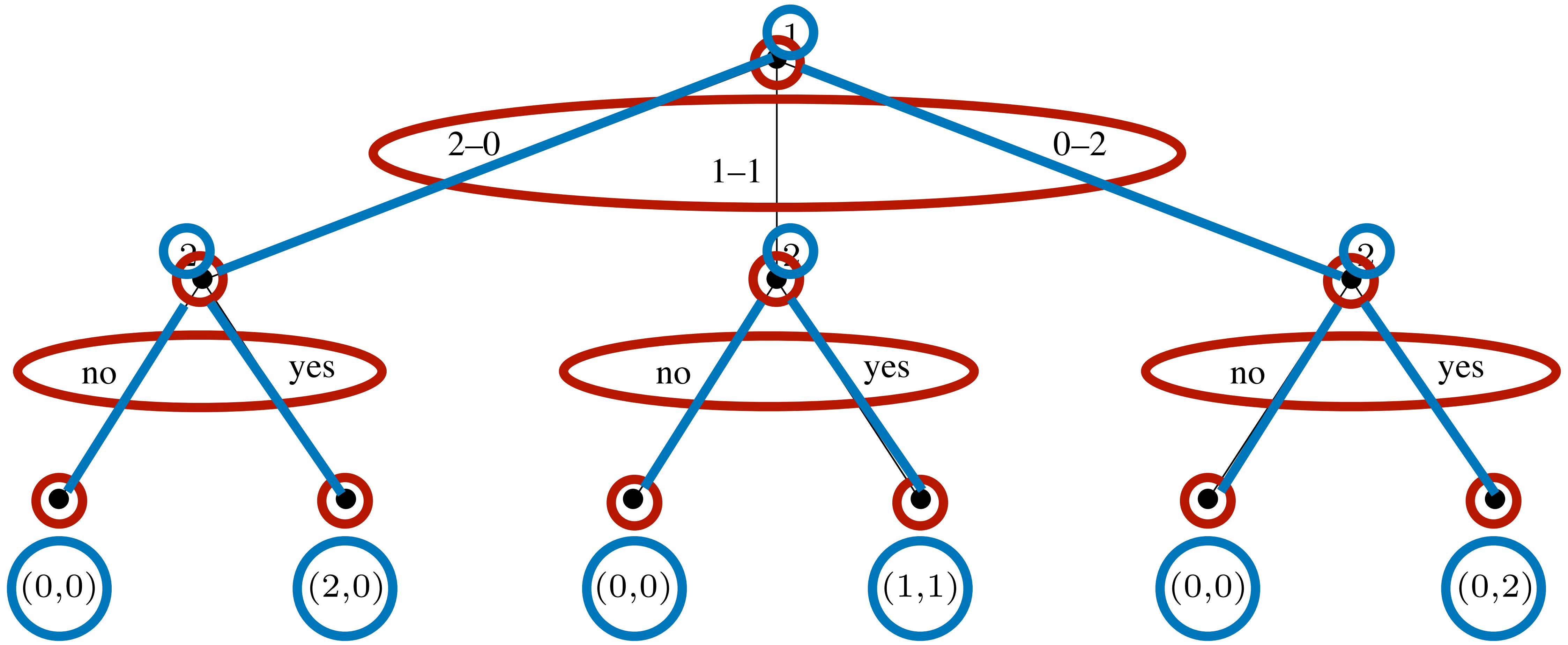
Key points:

1. If player i is mixing between two strategies in equilibrium, then they must **both** be **best responses**
2. Whether two strategies are best responses for i depends upon the probabilities that the **other player** plays their strategies

Recap: Perfect Information Extensive Form Game

Definition:
A finite perfect-information game in extensive form is a tuple $G = (N, A, H, Z, \chi, \rho, \sigma, u)$, where

- N is a set of n **players**,
- A is a single set of **actions**,
- H is a set of nonterminal **choice nodes**,
- Z is a set of **terminal nodes** (disjoint from H),
- $\chi : H \to 2^A$ is the **action function**,
- $\rho : H \to N$ is the **player function**,
- $\sigma : H \times A \to H \cup Z$ is the **successor function**,
- $u = (u_1, u_2, \ldots, u_n)$ is a profile of **utility functions** for each player, with $u_i : Z \to \mathbb{R}$.



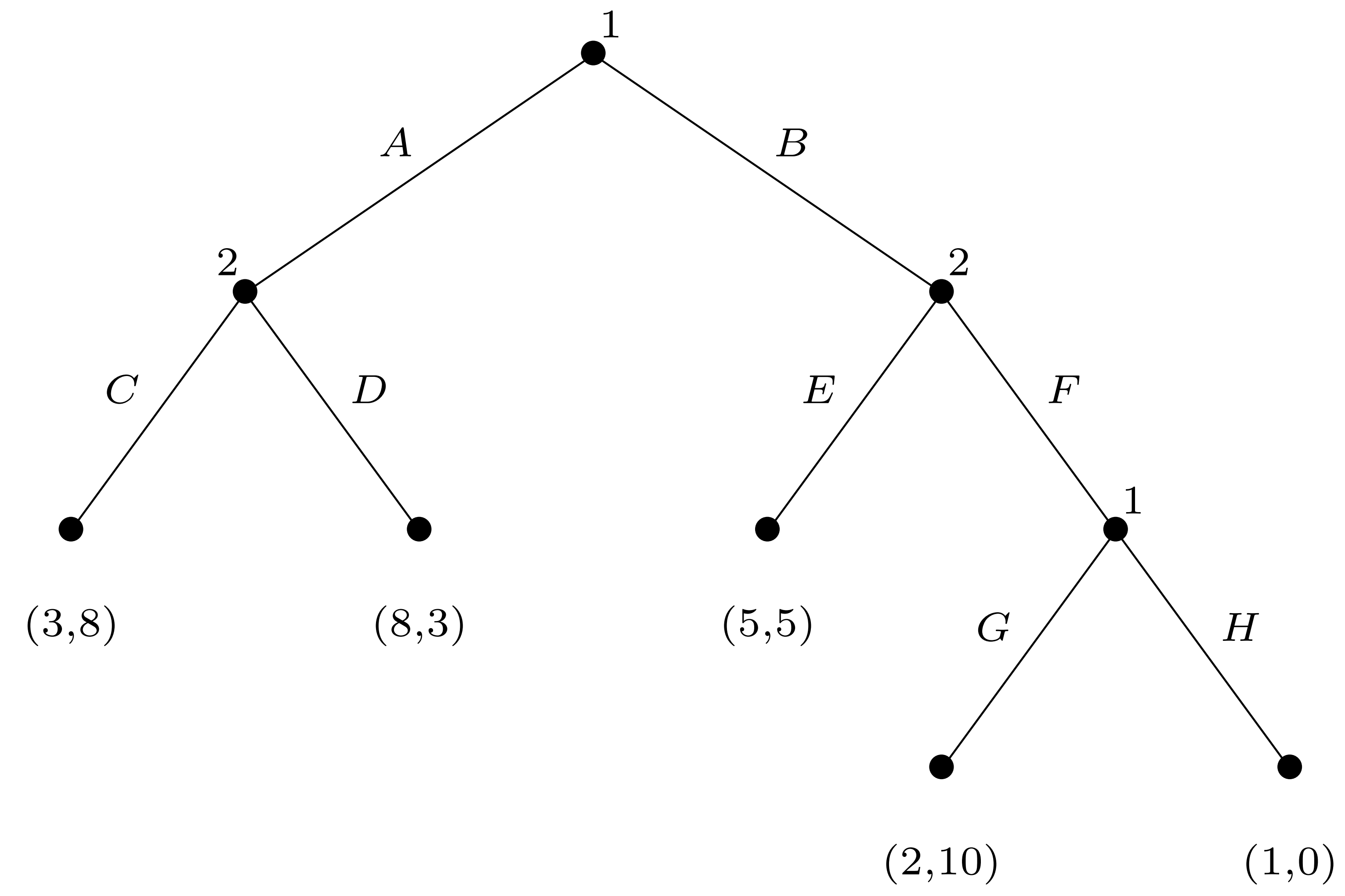
Recap: Pure Strategies

Definition:
Let $G = (N, A, H, Z, \chi, \rho, \sigma, u)$ be a perfect information game in extensive form. Then the **pure strategies of player i** consist of the cross product of actions available to player i at each of their choice nodes, i.e.,

$$\prod_{h \in H | \rho(h) = i} \chi(h).$$

Note: A pure strategy associates an action with **each** choice node, even those that will **never be reached.**

Recap: Induced Normal Form

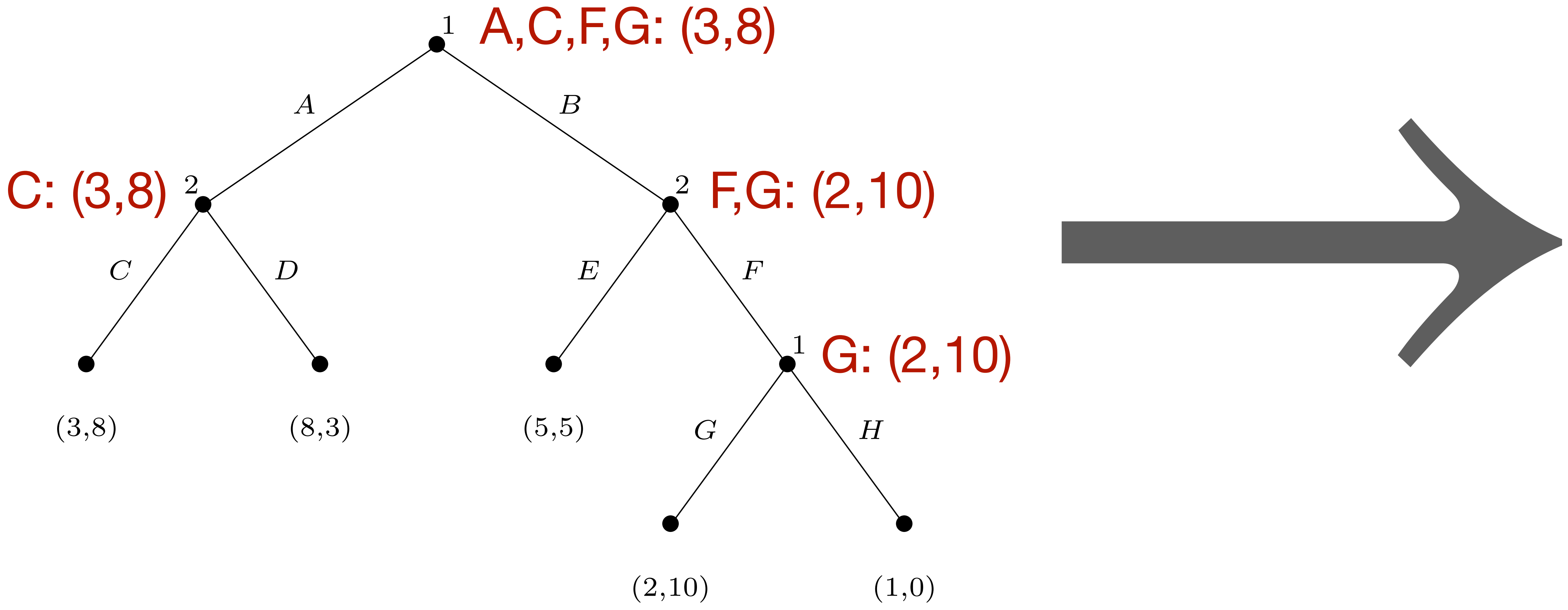


	C,E	C,F	D,E	D,F
A,G	3,8	3,8	8,3	8,3
A,H	3,8	3,8	8,3	8,3
B,G	5,5	2,10	5,5	2,10
B,H	5,5	1,0	5,5	1,0

- Any pair of pure strategies uniquely identifies a **terminal node**, which identifies a **utility** for each agent
- We have now defined a set of **agents**, **pure strategies**, and **utility functions**
- Any extensive form game defines a corresponding **induced normal form game**

Recap: Backward Induction

- **Backward induction** is a straightforward algorithm that is guaranteed to compute a subgame perfect equilibrium.
- **Idea:** Replace subgames lower in the tree with their equilibrium values



$$(A, G), (C, F)$$

Lecture Outline

1. Hints & Recap
2. Imperfect Information Games
3. Behavioural vs. Mixed Strategies
4. Perfect vs. Imperfect Recall
5. Computational Issues

Imperfect Information, informally

- **Perfect information** games model **sequential** actions that are **observed by all players**

Imperfect Information, informally

- **Perfect information** games model **sequential** actions that are **observed by all players**
  - **Randomness** can be modelled by a special *Nature* player with constant utility

Imperfect Information, informally

- **Perfect information** games model **sequential** actions that are **observed by all players**
    - **Randomness** can be modelled by a special *Nature* player with constant utility
- But many games involve **hidden** actions

Imperfect Information, informally

- **Perfect information** games model **sequential** actions that are **observed by all players**
    - **Randomness** can be modelled by a special *Nature* player with constant utility
- But many games involve **hidden** actions
    - Cribbage, poker, Scrabble

Imperfect Information, informally

- **Perfect information** games model **sequential** actions that are **observed by all players**
  - **Randomness** can be modelled by a special *Nature* player with constant utility
- But many games involve **hidden** actions
  - Cribbage, poker, Scrabble
  - Sometimes actions of the **players** are hidden, sometimes ***Nature*'s** actions are hidden, sometimes both

Imperfect Information, informally

- **Perfect information** games model **sequential** actions that are **observed by all players**
  - **Randomness** can be modelled by a special *Nature* player with constant utility
- But many games involve **hidden** actions
  - Cribbage, poker, Scrabble
  - Sometimes actions of the **players** are hidden, sometimes ***Nature*'s** actions are hidden, sometimes both
- **Imperfect information extensive form games** are a model of games with sequential actions, some of which may be **hidden**

Imperfect Information Extensive Form Game

Definition:

An **imperfect information game in extensive form** is a tuple $G = (N, A, H, Z, \chi, \rho, \sigma, u, I)$, where

Imperfect Information Extensive Form Game

Definition:
An **imperfect information game in extensive form** is a tuple $G = (N, A, H, Z, \chi, \rho, \sigma, u, I)$, where

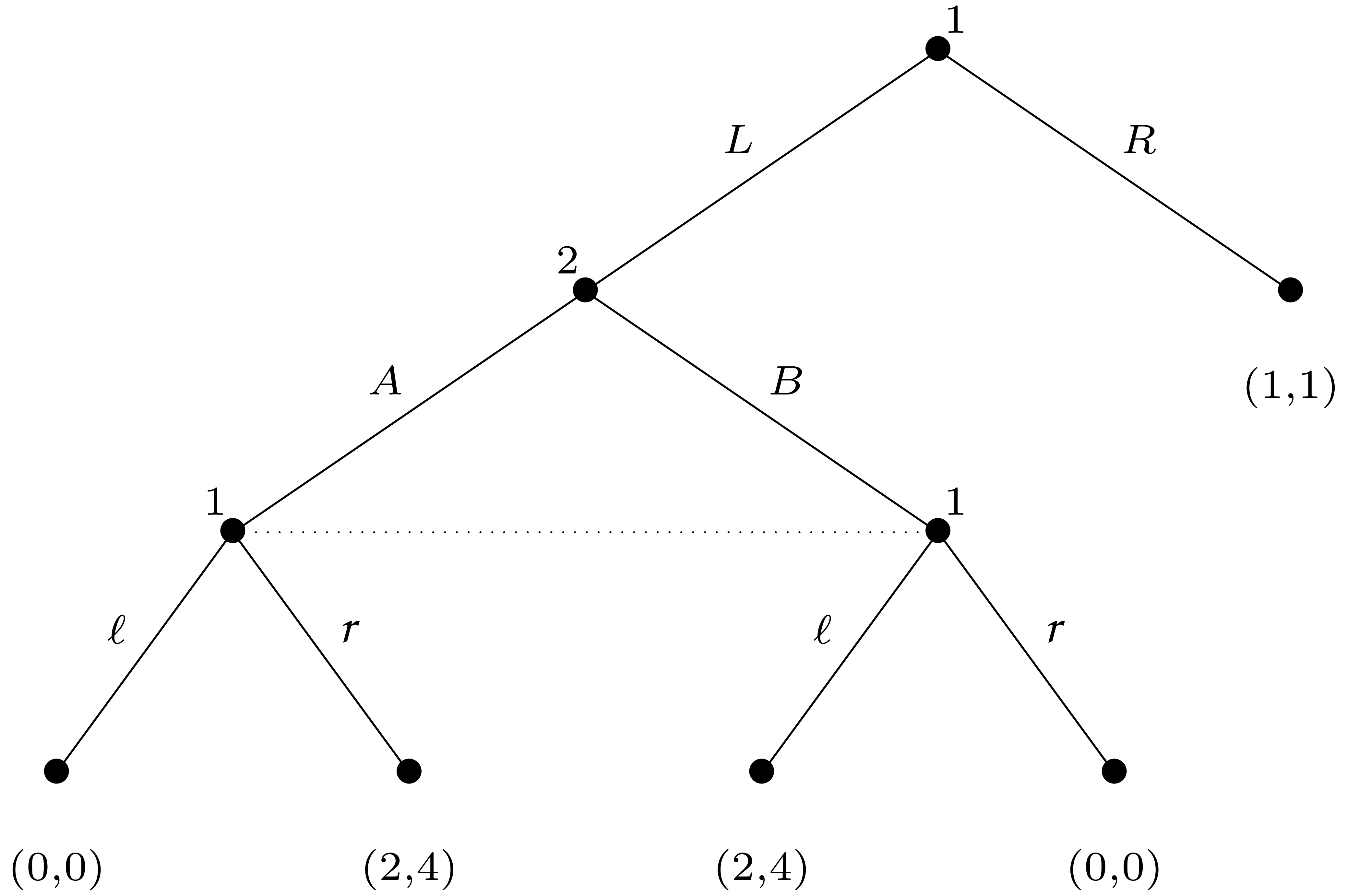
- $(N, A, H, Z, \chi, \rho, \sigma, u)$ is a perfect information extensive form game, and

Imperfect Information Extensive Form Game

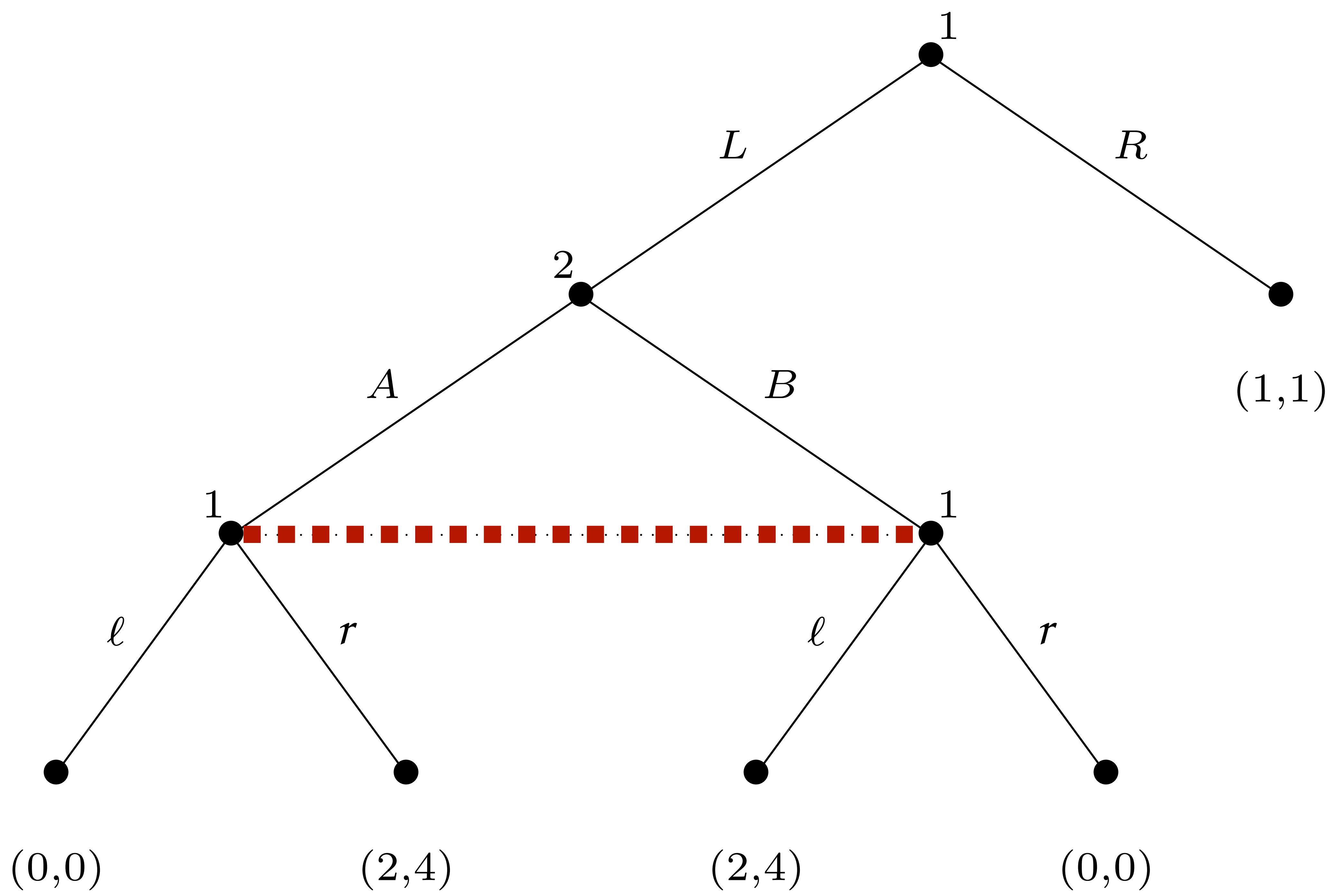
Definition:
An **imperfect information game in extensive form** is a tuple $G = (N, A, H, Z, \chi, \rho, \sigma, u, I)$, where

- $(N, A, H, Z, \chi, \rho, \sigma, u)$ is a perfect information extensive form game, and
- $I = (I_1, \ldots, I_n)$, where $I_i = (I_{i,1}, \ldots, I_{i,k_i})$ is an **equivalence relation** on (i.e., partition of) $\{h \in H : \rho(h) = i\}$ with the property that $\chi(h) = \chi(h')$ and $\rho(h) = \rho(h')$ whenever there exists a j for which $h \in I_{i,j}$ and $h' \in I_{i,j}$.

Imperfect Information Extensive Form Example

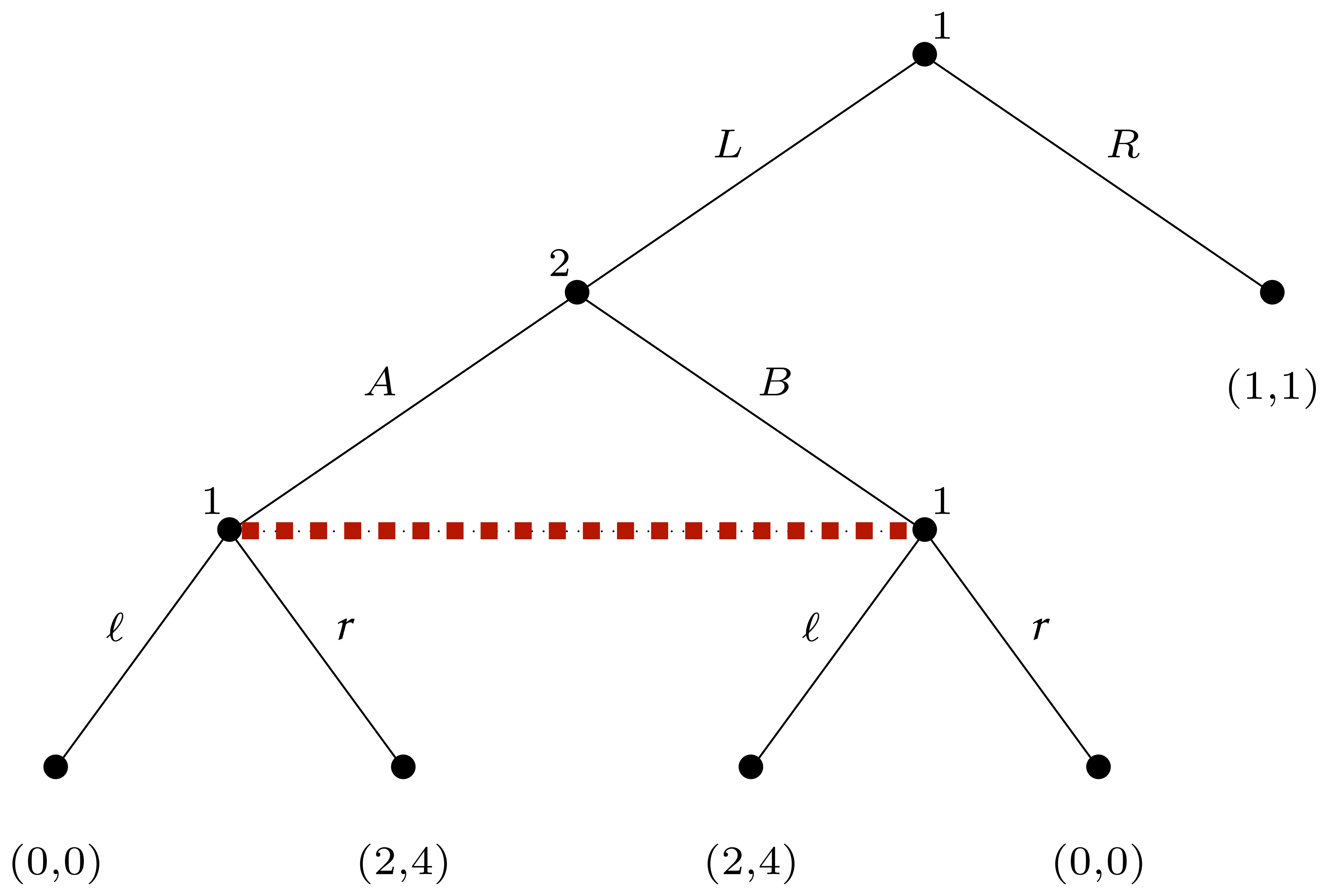


Imperfect Information Extensive Form Example



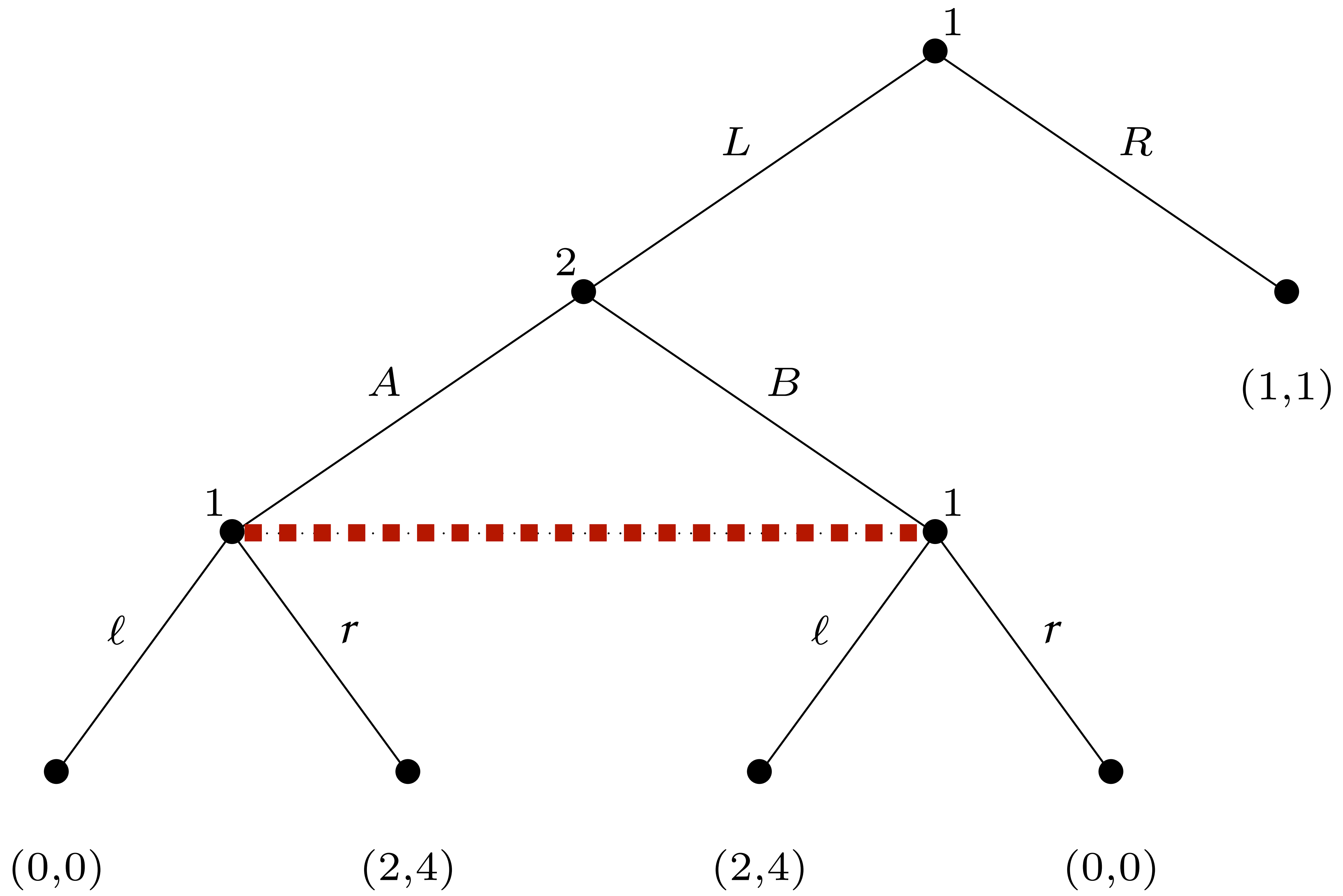
- The elements of the partition are sometimes called **information sets**

Imperfect Information Extensive Form Example



- The elements of the partition are sometimes called **information sets**
- Players **cannot distinguish** which **history** they are in within an information set

Imperfect Information Extensive Form Example



- The elements of the partition are sometimes called **information sets**
- Players **cannot distinguish** which **history** they are in within an information set
- **Question:** What are the information sets for each player in this game?

Pure Strategies

Pure Strategies

Question: What are the **pure strategies** in an **imperfect information** extensive form game?

Pure Strategies

Question: What are the **pure strategies** in an **imperfect information** extensive form game?

Definition:
Let $G = (N, A, H, Z, \chi, \rho, \sigma, u, I)$ be an imperfect information game in extensive form. Then the **pure strategies of player *i*** consist of the cross product of actions available to player *i* at each of their **information sets**, i.e.,

$$\prod_{I_{i,j} \in I_i} \chi(h)$$

- A pure strategy associates an action with **each** information set, even those that will **never be reached**

Pure Strategies

Question: What are the **pure strategies** in an **imperfect information** extensive form game?

Definition:
Let $G = (N, A, H, Z, \chi, \rho, \sigma, u, I)$ be an imperfect information game in extensive form. Then the **pure strategies of player *i*** consist of the cross product of actions available to player *i* at each of their **information sets**, i.e.,

$$\prod_{I_{i,j} \in I_i} \chi(h)$$

- A pure strategy associates an action with **each** information set, even those that will **never be reached**

Questions:

In an imperfect information game:

Pure Strategies

Question: What are the **pure strategies** in an **imperfect information** extensive form game?

Definition:
Let $G = (N, A, H, Z, \chi, \rho, \sigma, u, I)$ be an imperfect information game in extensive form. Then the **pure strategies of player *i*** consist of the cross product of actions available to player *i* at each of their **information sets**, i.e.,

$$\prod_{I_{i,j} \in I_i} \chi(h)$$

- A pure strategy associates an action with **each** information set, even those that will **never be reached**

Questions:

In an imperfect information game:

1. What are the **mixed strategies**?

Pure Strategies

Question: What are the **pure strategies** in an **imperfect information** extensive form game?

Definition:
Let $G = (N, A, H, Z, \chi, \rho, \sigma, u, I)$ be an imperfect information game in extensive form. Then the **pure strategies of player *i*** consist of the cross product of actions available to player *i* at each of their **information sets**, i.e.,

$$\prod_{I_{i,j} \in I_i} \chi(h)$$

- A pure strategy associates an action with **each** information set, even those that will **never be reached**

Questions:

In an imperfect information game:

1. What are the **mixed strategies**?
2. What is a **best response**?

Pure Strategies

Question: What are the **pure strategies** in an **imperfect information** extensive form game?

Definition:
Let $G = (N, A, H, Z, \chi, \rho, \sigma, u, I)$ be an imperfect information game in extensive form. Then the **pure strategies of player *i*** consist of the cross product of actions available to player *i* at each of their **information sets**, i.e.,

$$\prod_{I_{i,j} \in I_i} \chi(h)$$

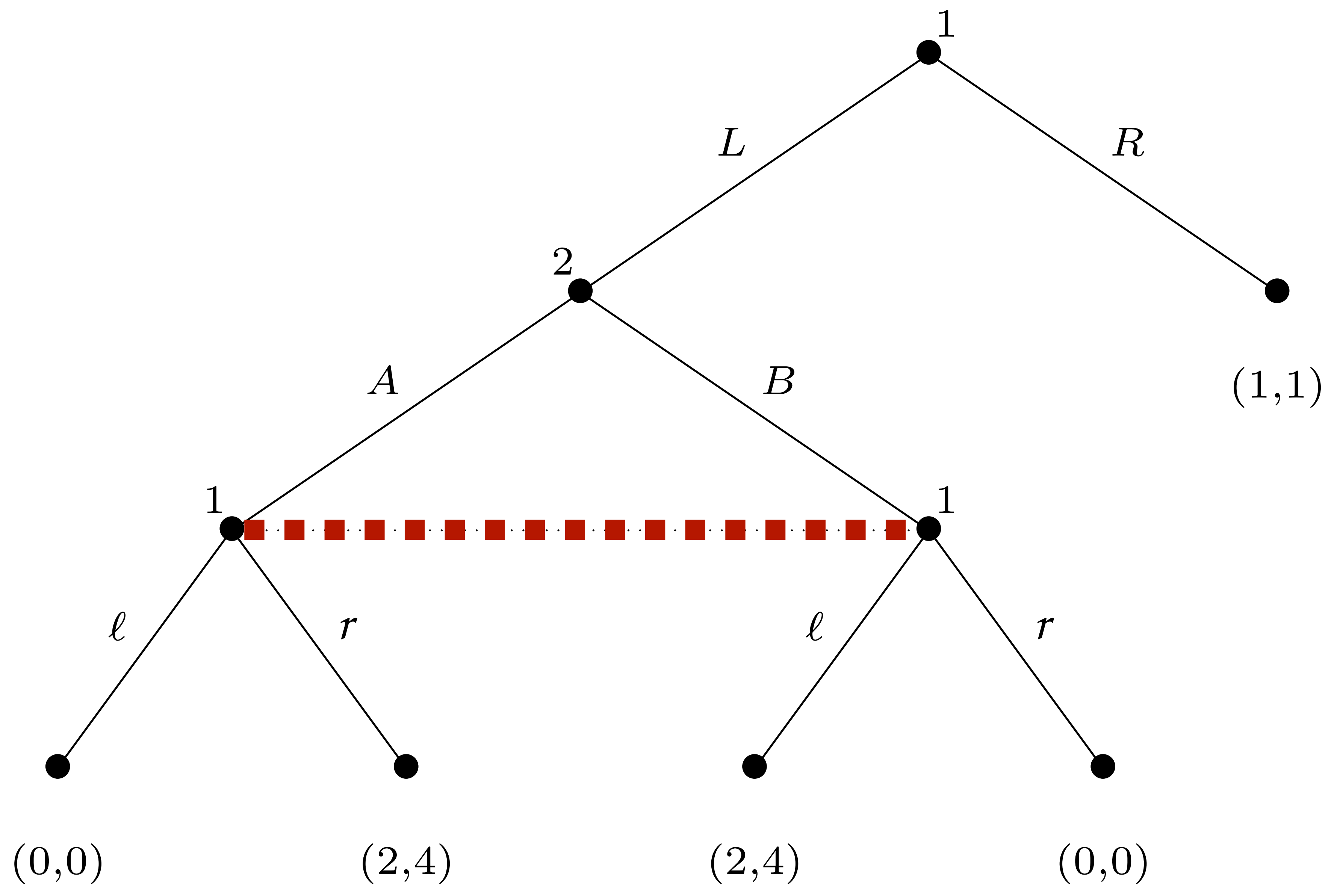
- A pure strategy associates an action with **each** information set, even those that will **never be reached**

Questions:

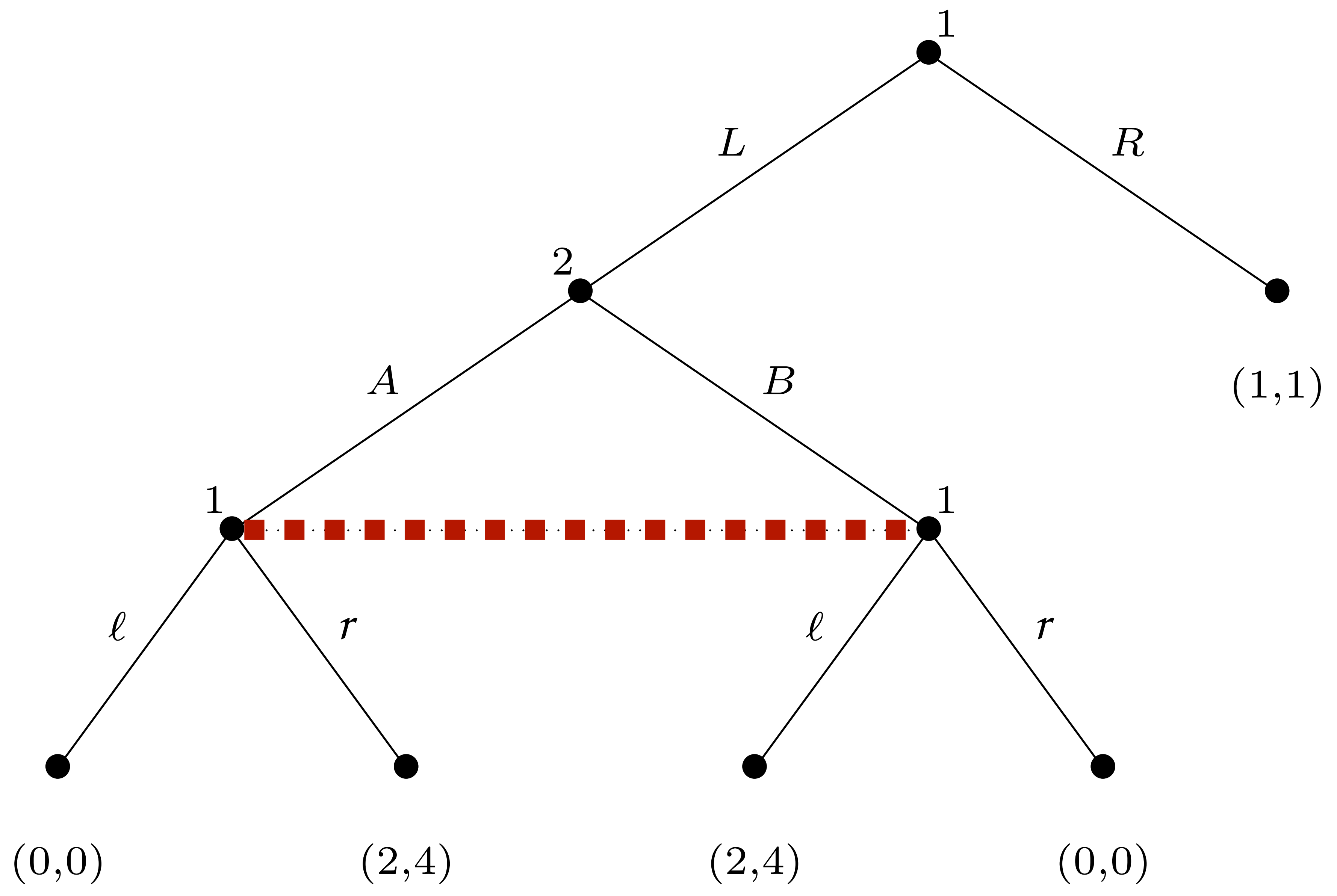
In an imperfect information game:

1. What are the **mixed strategies**?
2. What is a **best response**?
3. What is a **Nash equilibrium**?

Induced Normal Form

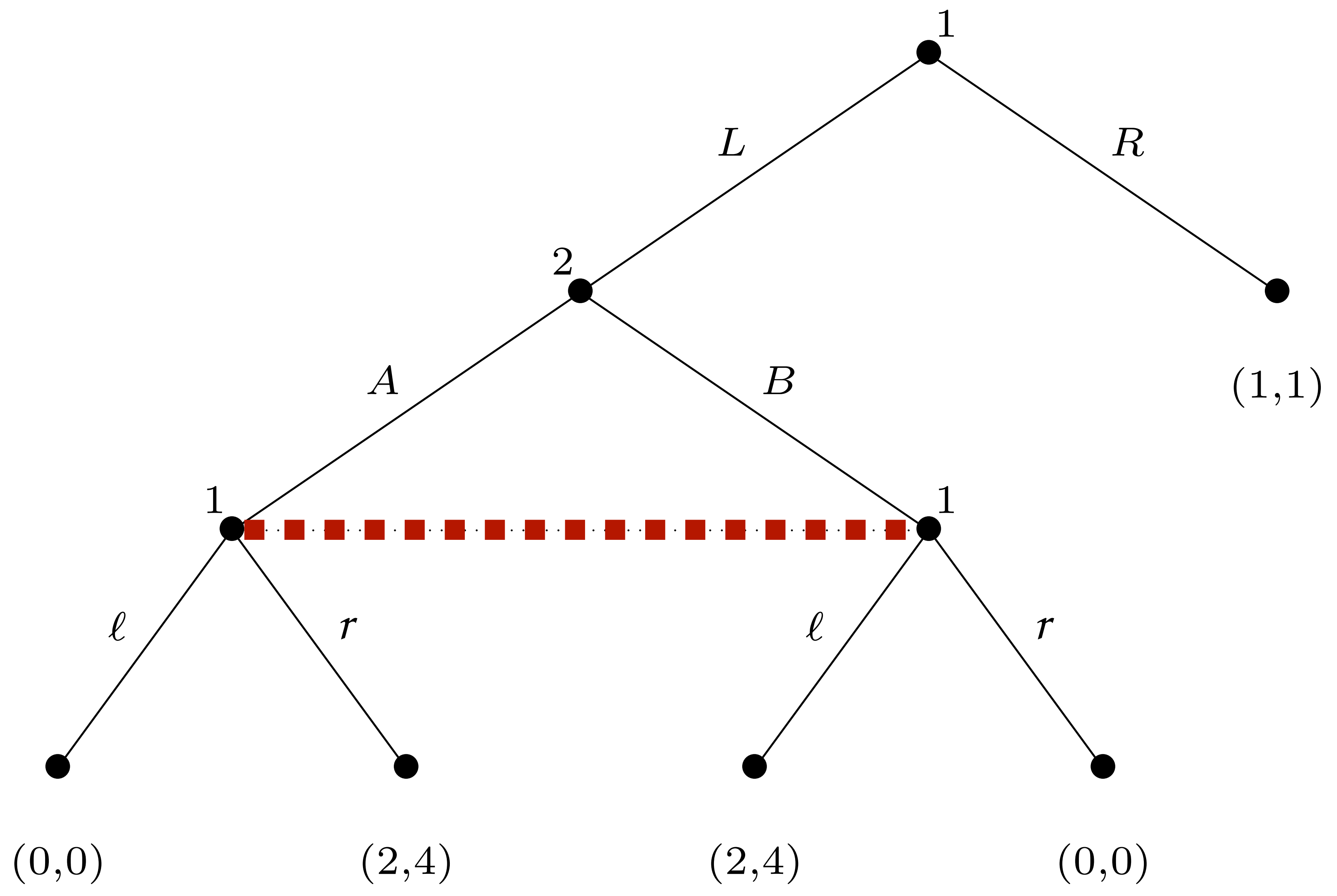


Induced Normal Form



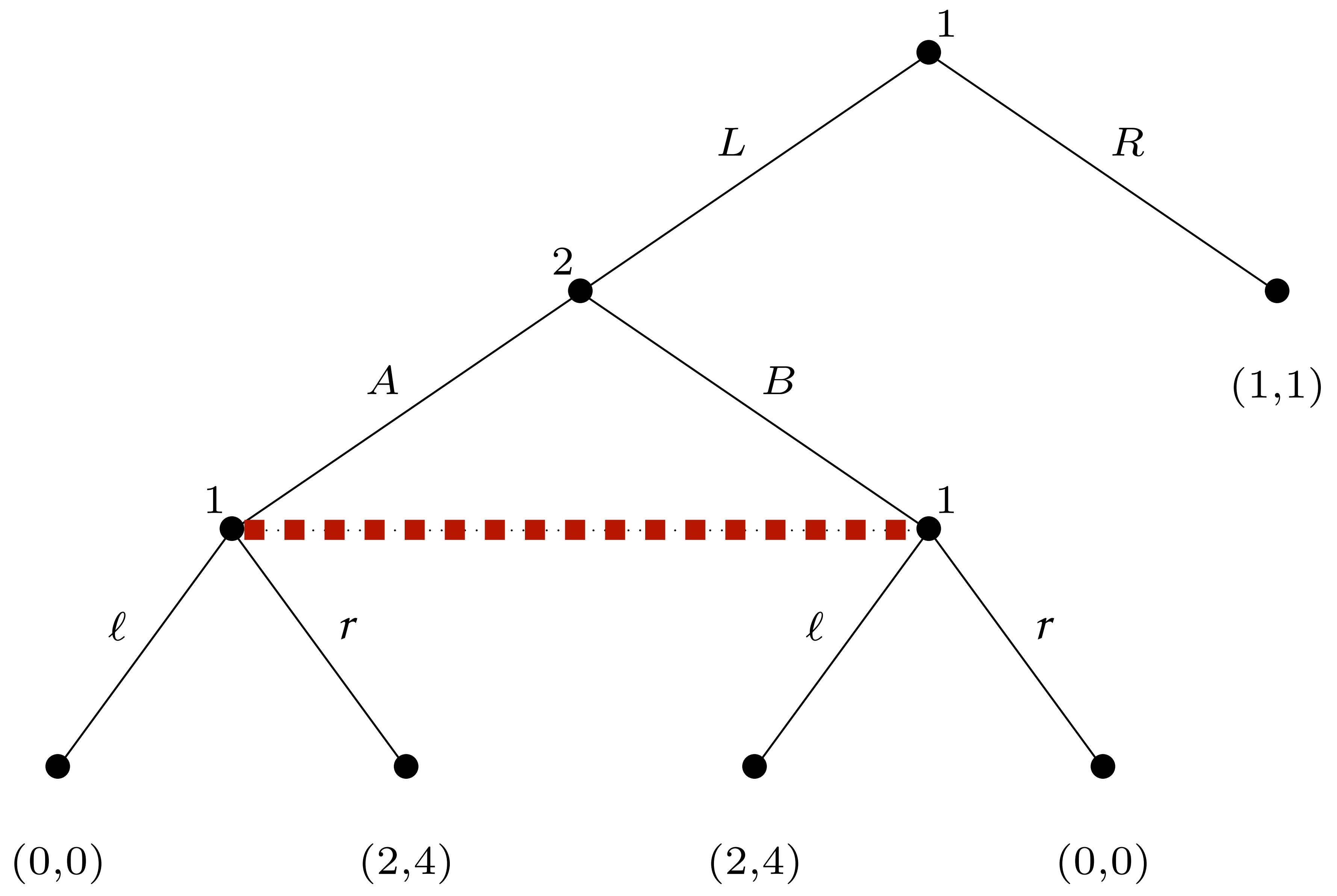
- Any pair of pure strategies uniquely identifies a **terminal node**, which identifies a **utility** for each agent

Induced Normal Form



- Any pair of pure strategies uniquely identifies a **terminal node**, which identifies a **utility** for each agent
- We have now defined a set of **agents**, **pure strategies**, and **utility functions**

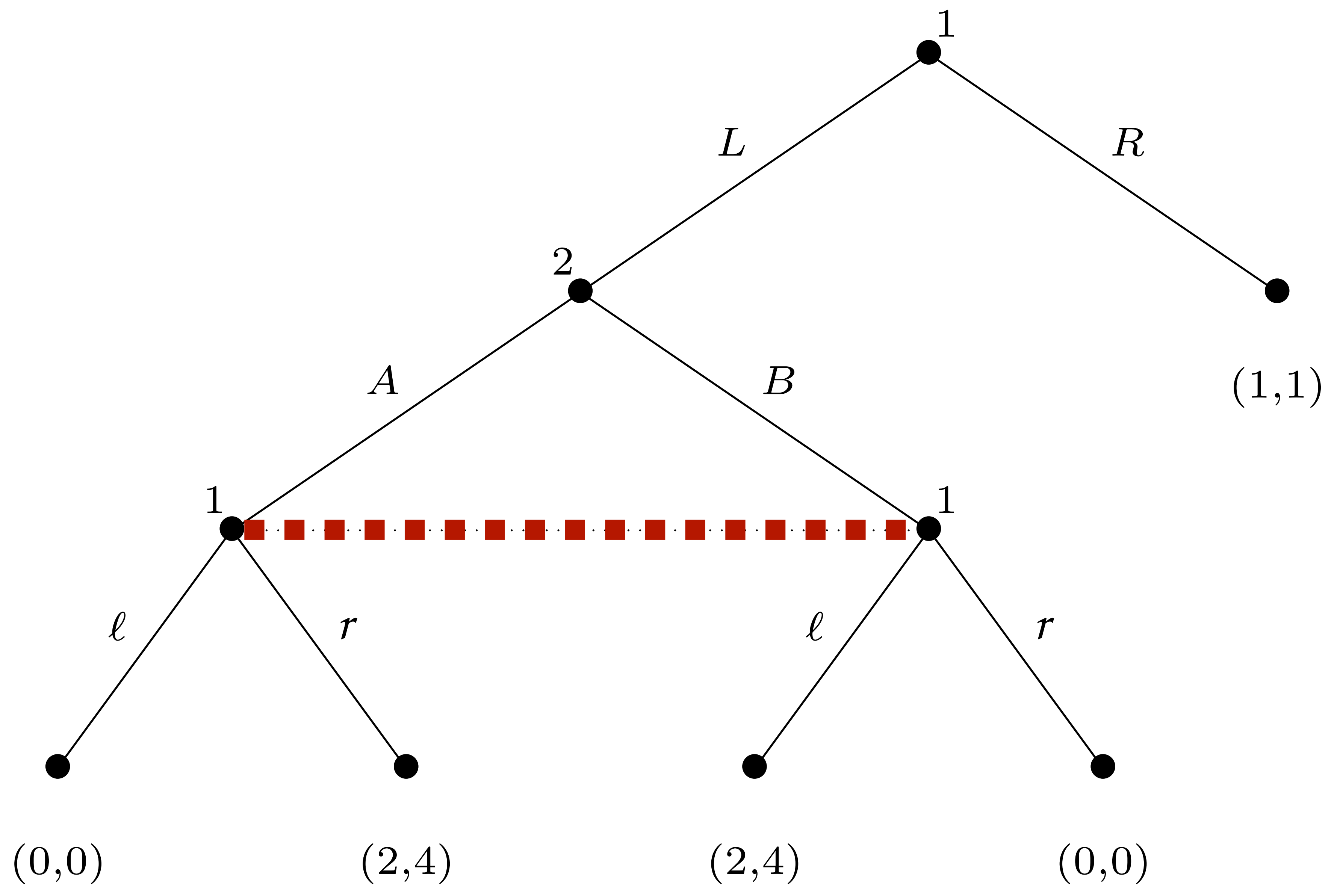
Induced Normal Form



	A	B
L,ℓ	0,0	2,4
L,r	2,4	0,0
R,ℓ	1,1	1,1
R,r	1,1	1,1

- Any pair of pure strategies uniquely identifies a **terminal node**, which identifies a **utility** for each agent
- We have now defined a set of **agents**, **pure strategies**, and **utility functions**
- Any extensive form game defines a corresponding **induced normal form game**

Induced Normal Form



	A	B
L,ℓ	0,0	2,4
L,r	2,4	0,0
R,ℓ	1,1	1,1
R,r	1,1	1,1

Question:
Can you represent an arbitrary **perfect information** extensive form game as an **imperfect information** game?

- Any pair of pure strategies uniquely identifies a **terminal node**, which identifies a **utility** for each agent
- We have now defined a set of **agents**, **pure strategies**, and **utility functions**
- Any extensive form game defines a corresponding **induced normal form game**

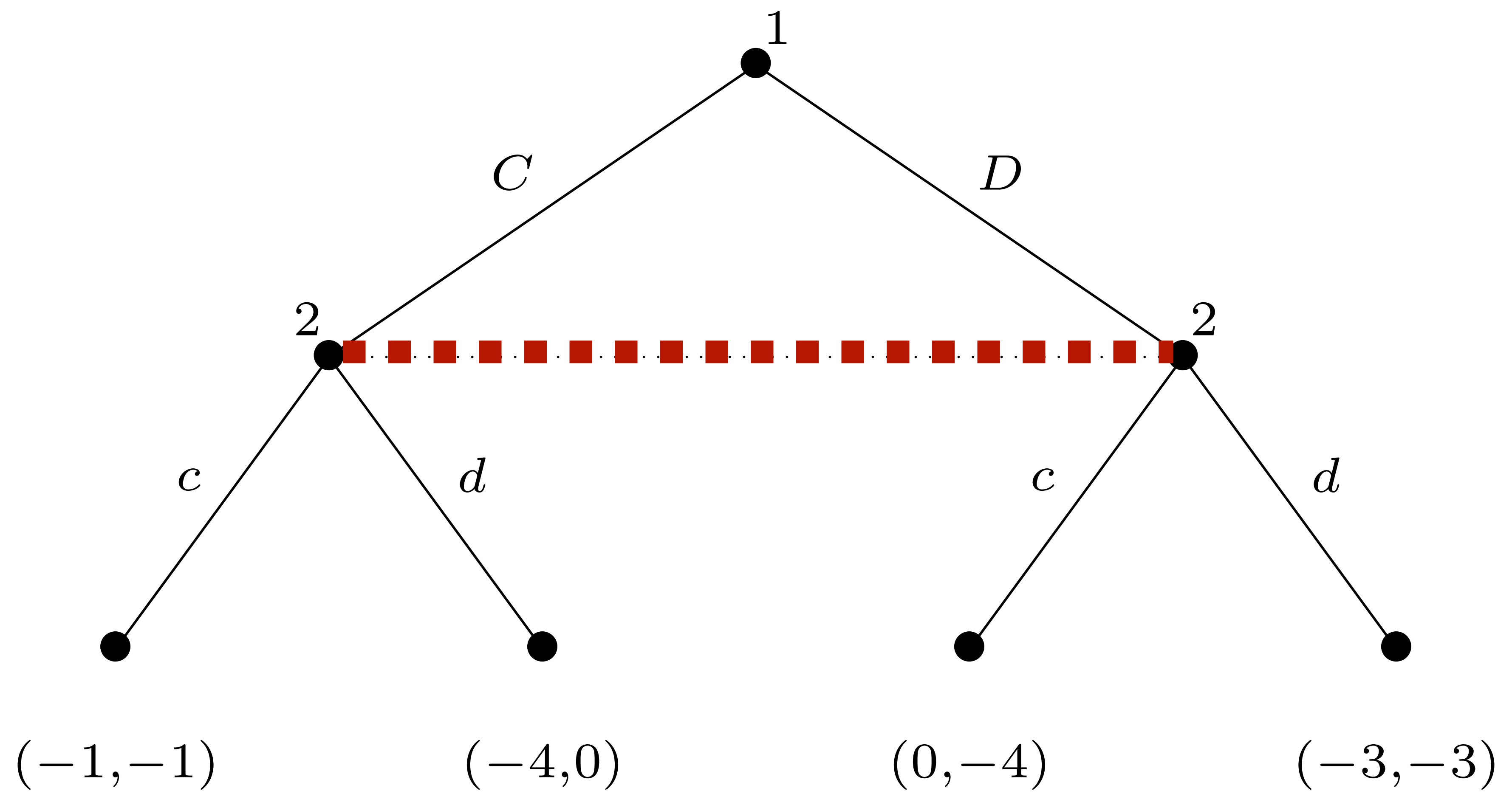
Normal to Extensive Form

	c	d
C	-1,-1	-4,0
D	0,-4	-3,-3

- Unlike perfect information games, we can go in the opposite direction and represent **any normal form game** as an **imperfect information extensive form game**

Normal to Extensive Form

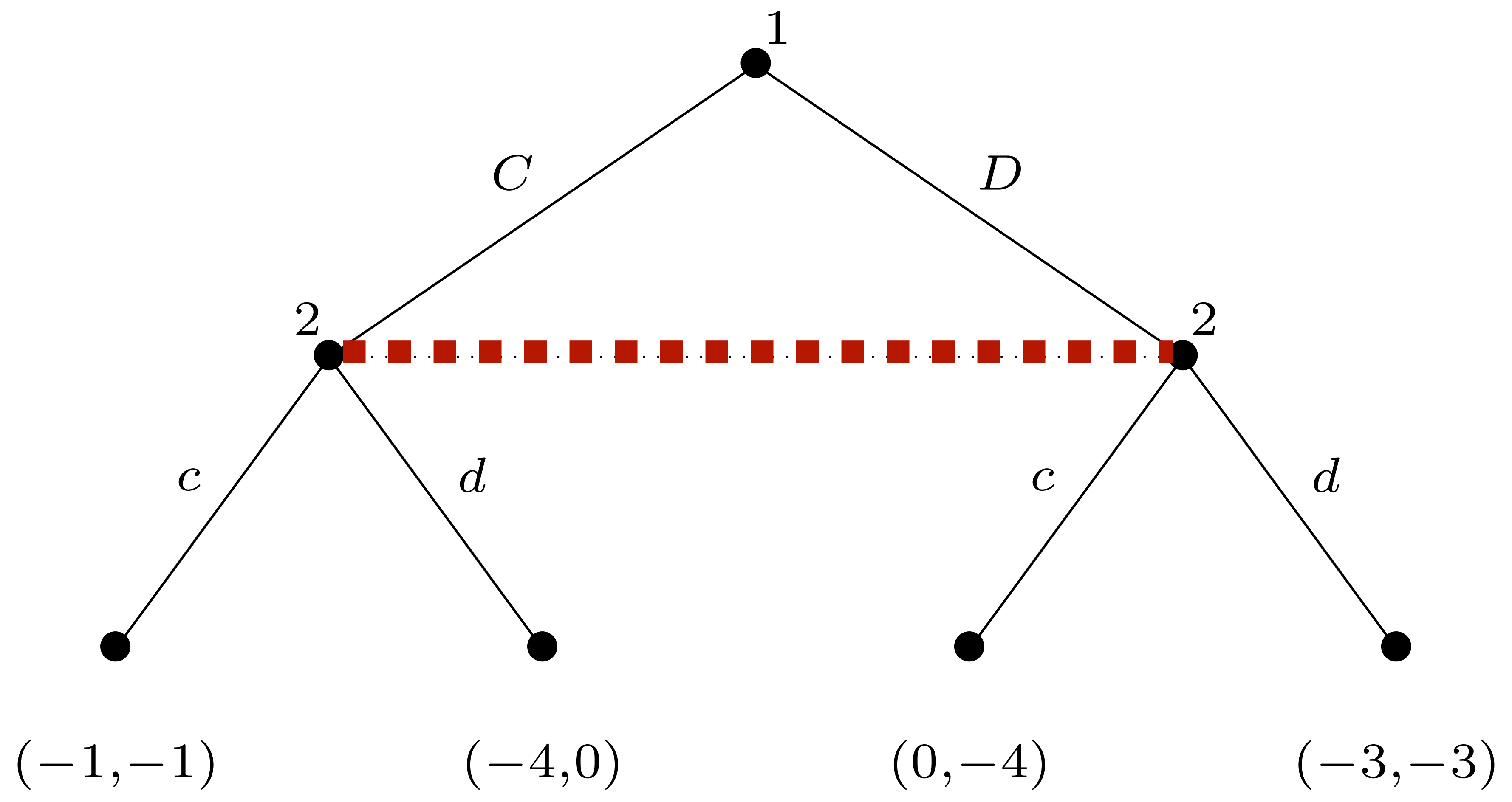
	c	d
C	-1,-1	-4,0
D	0,-4	-3,-3



- Unlike perfect information games, we can go in the opposite direction and represent **any normal form game** as an **imperfect information extensive form game**

Normal to Extensive Form

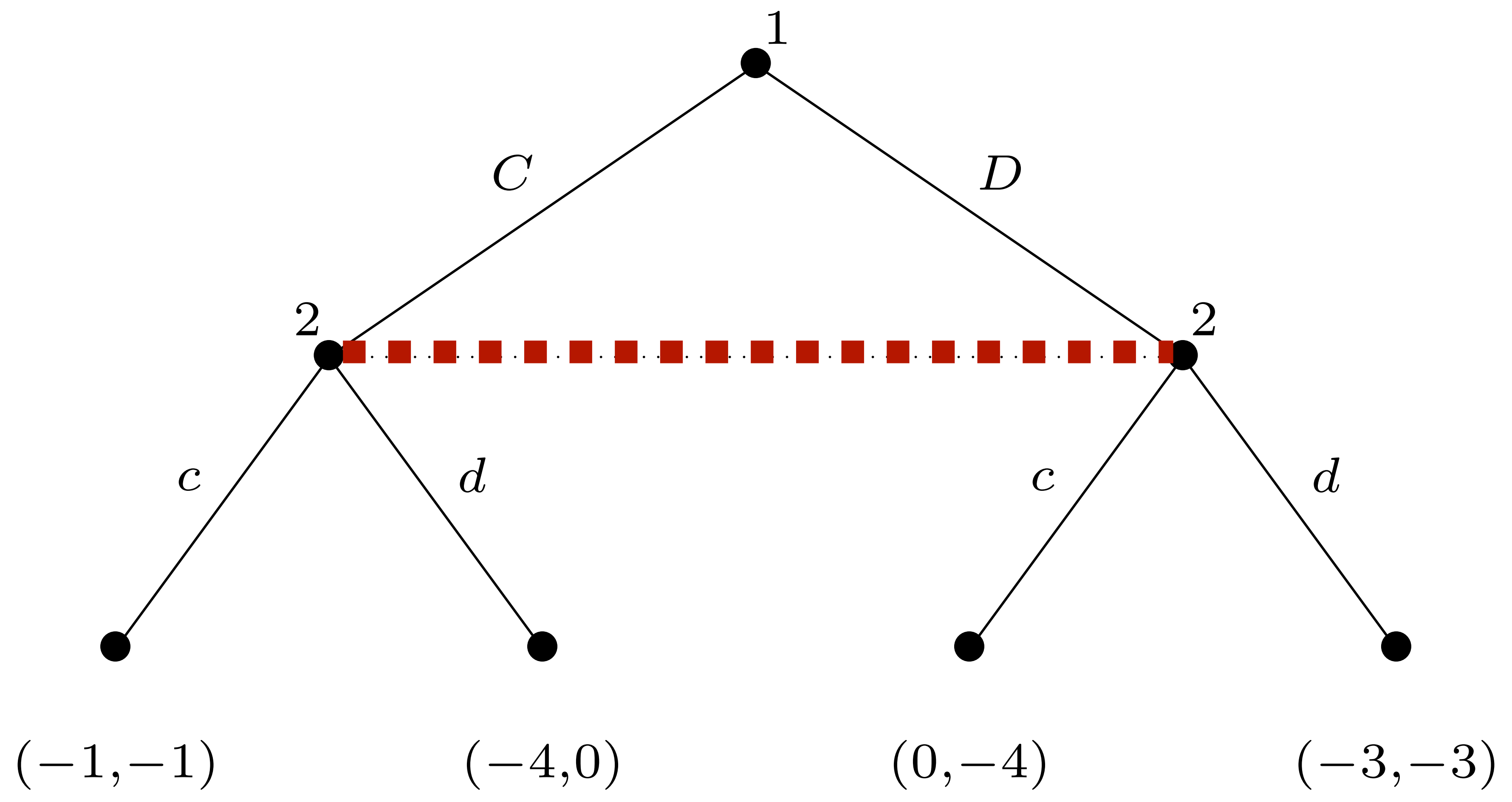
	c	d
C	-1,-1	-4,0
D	0,-4	-3,-3



- Unlike perfect information games, we can go in the opposite direction and represent **any normal form game** as an **imperfect information extensive form game**
- Players can play in **any order** (**why?**)

Normal to Extensive Form

	c	d
C	-1,-1	-4,0
D	0,-4	-3,-3



- Unlike perfect information games, we can go in the opposite direction and represent **any normal form game** as an **imperfect information extensive form game**
- Players can play in **any order** (**why?**)
- **Question:** What happens if we run this translation on the induced normal form?

Behavioural vs.
Mixed Strategies

Behavioural vs. Mixed Strategies

Definition:
A **mixed strategy** $s_i \in \Delta(A^{I_i})$ is any distribution over an agent's **pure strategies**.

Behavioural vs. Mixed Strategies

Definition:

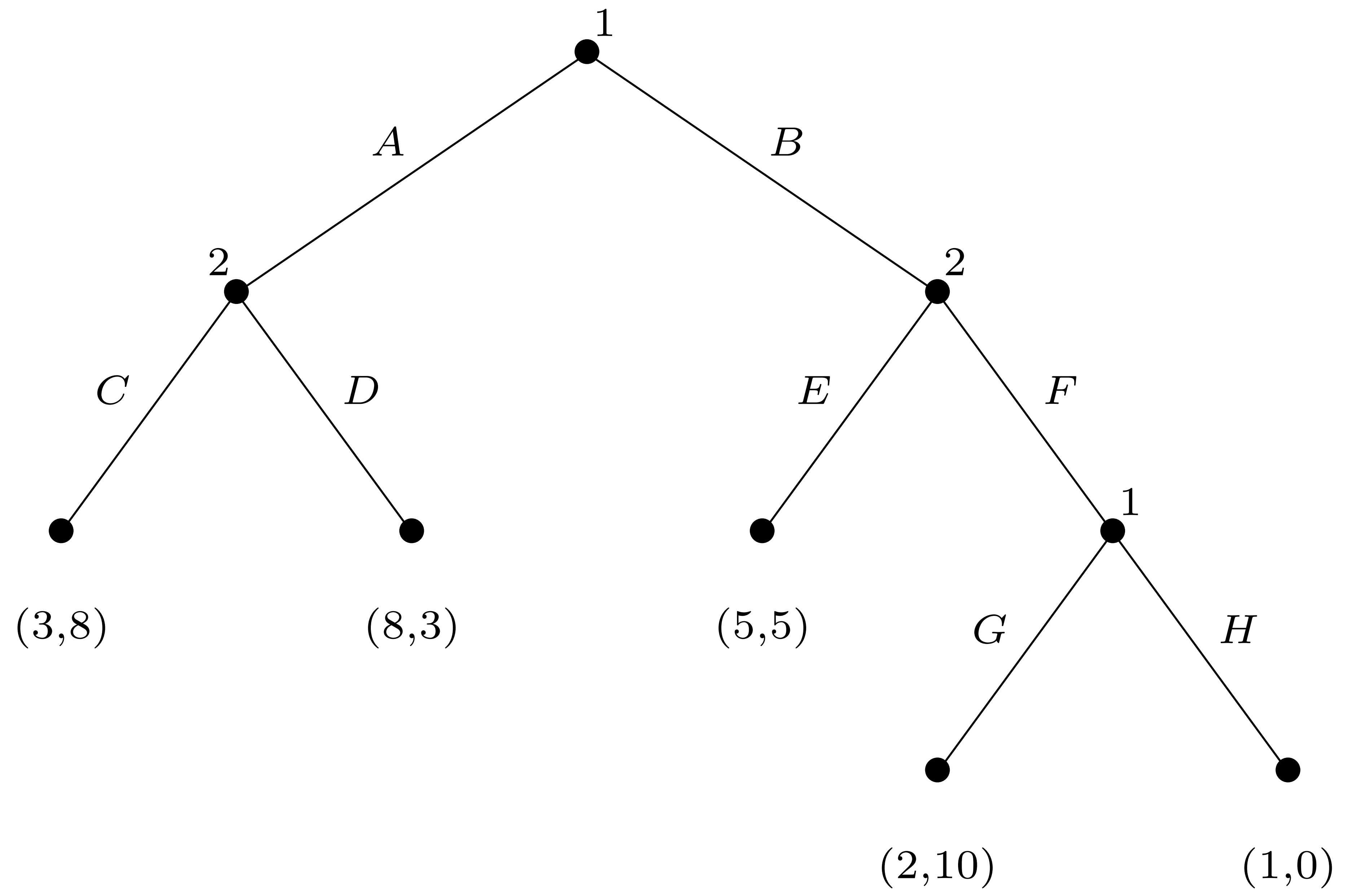
A **mixed strategy** $s_i \in \Delta(A^{I_i})$ is any distribution over an agent's **pure strategies**.

Definition:

A **behavioural strategy** $b_i \in [\Delta(A)]^{I_i}$ is a probability distribution over an agent's actions at an **information set**, which is **sampled independently** each time the agent arrives at the information set.

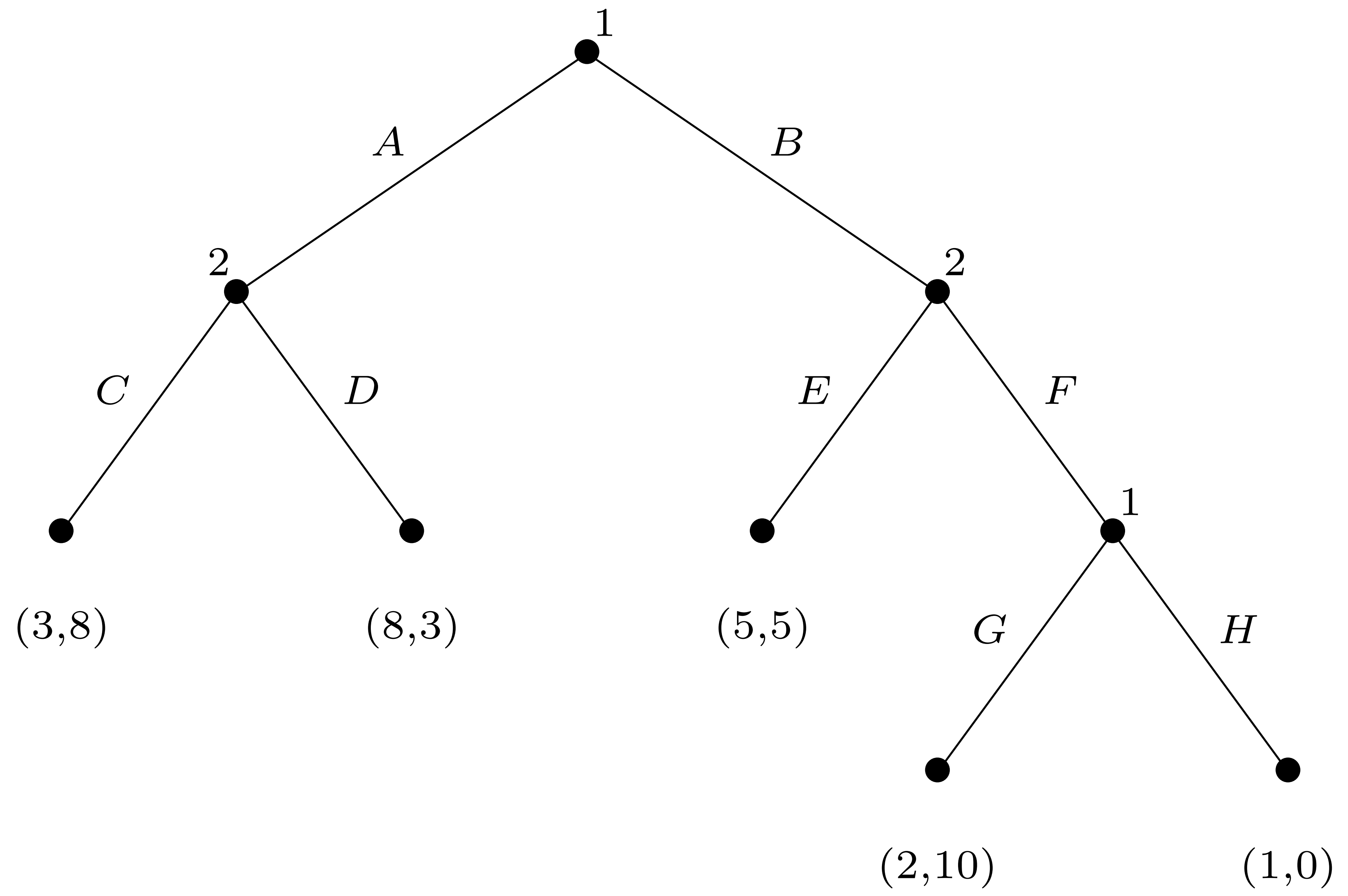
Behavioural vs. Mixed Example

- **Behavioural strategy**: ([.6:A, .4:B], [.6:G, .4:H])
- **Mixed strategy**: [.6:(A,G), .4:(B,H)]



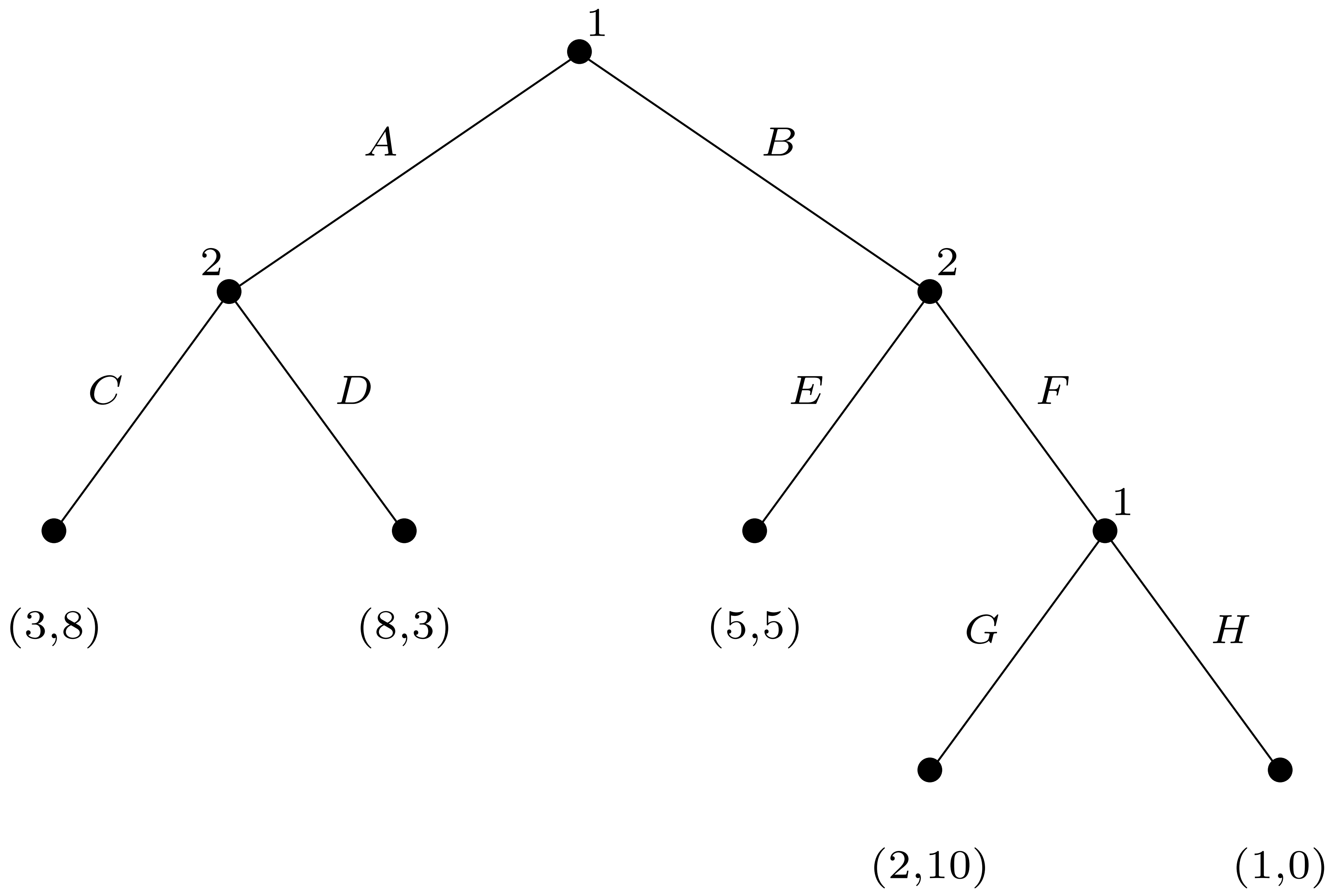
Behavioural vs. Mixed Example

- **Behavioural strategy**: ([.6:A, .4:B], [.6:G, .4:H])
- **Mixed strategy**: [.6:(A,G), .4:(B,H)]
- **Question:** Are these strategies **equivalent**? (**why**?)



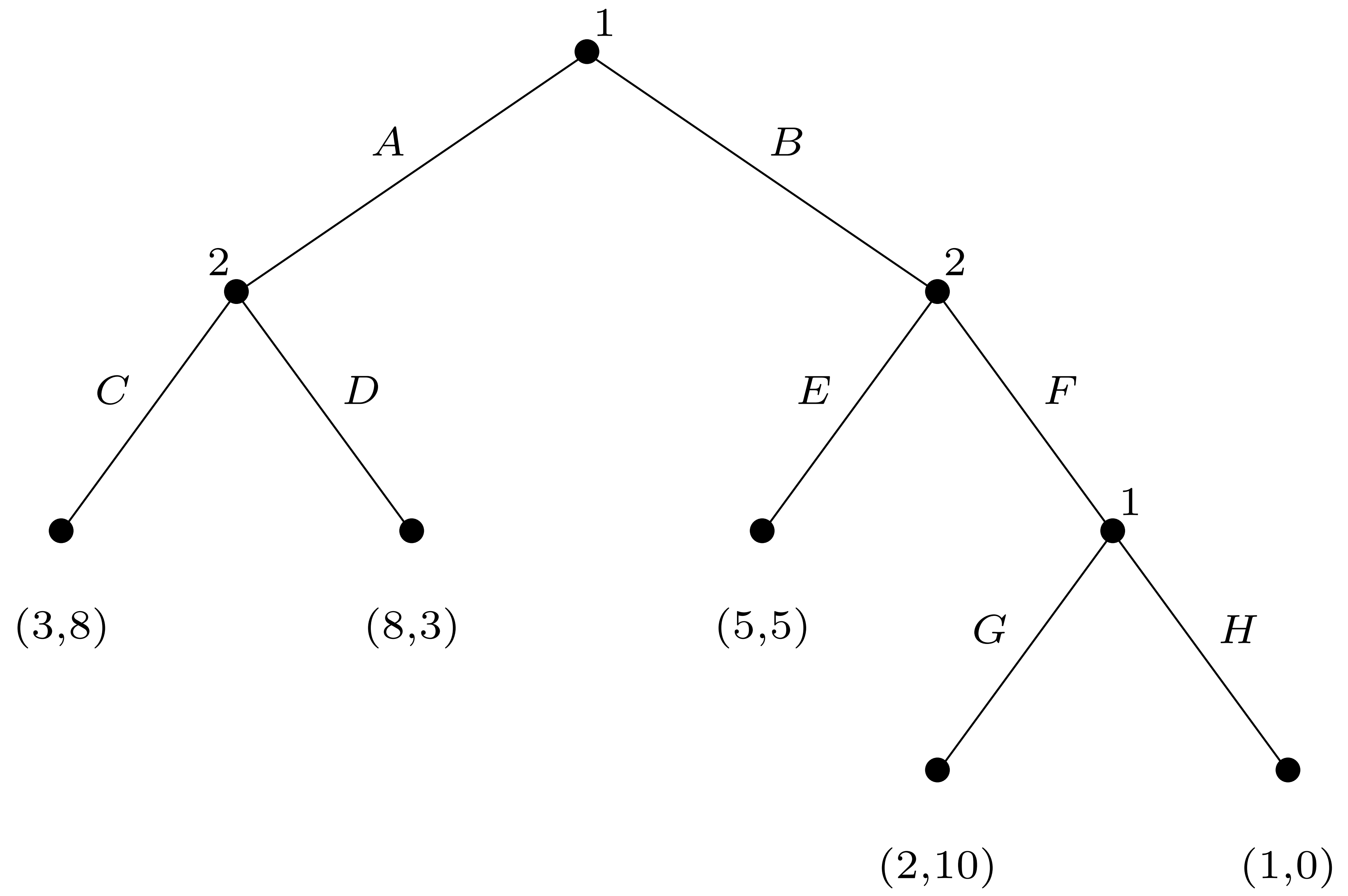
Behavioural vs. Mixed Example

- **Behavioural strategy**: ([.6:A, .4:B], [.6:G, .4:H])
- **Mixed strategy**: [.6:(A,G), .4:(B,H)]
- **Question:** Are these strategies **equivalent**? (**why**?)
- **Question:** Can you construct a **mixed strategy** that is equivalent to the behavioural strategy above?

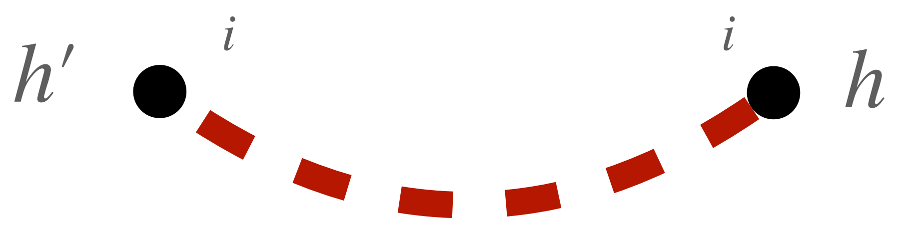


Behavioural vs. Mixed Example

- **Behavioural strategy**: ([.6:A, .4:B], [.6:G, .4:H])
- **Mixed strategy**: [.6:(A,G), .4:(B,H)]
- **Question:** Are these strategies **equivalent**? (**why**?)
- **Question:** Can you construct a **mixed strategy** that is equivalent to the behavioural strategy above?
- **Question:** Can you construct a **behavioural strategy** that is equivalent to the mixed strategy above?



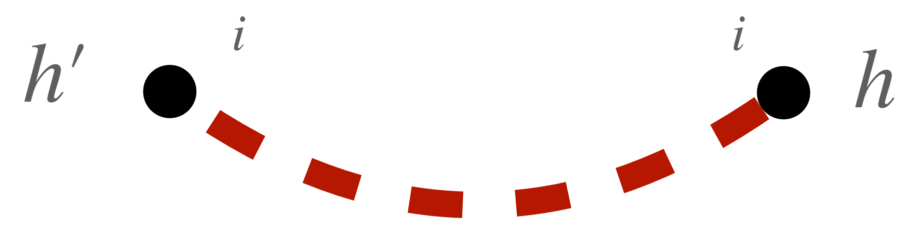
Perfect Recall



Perfect Recall

Definition:

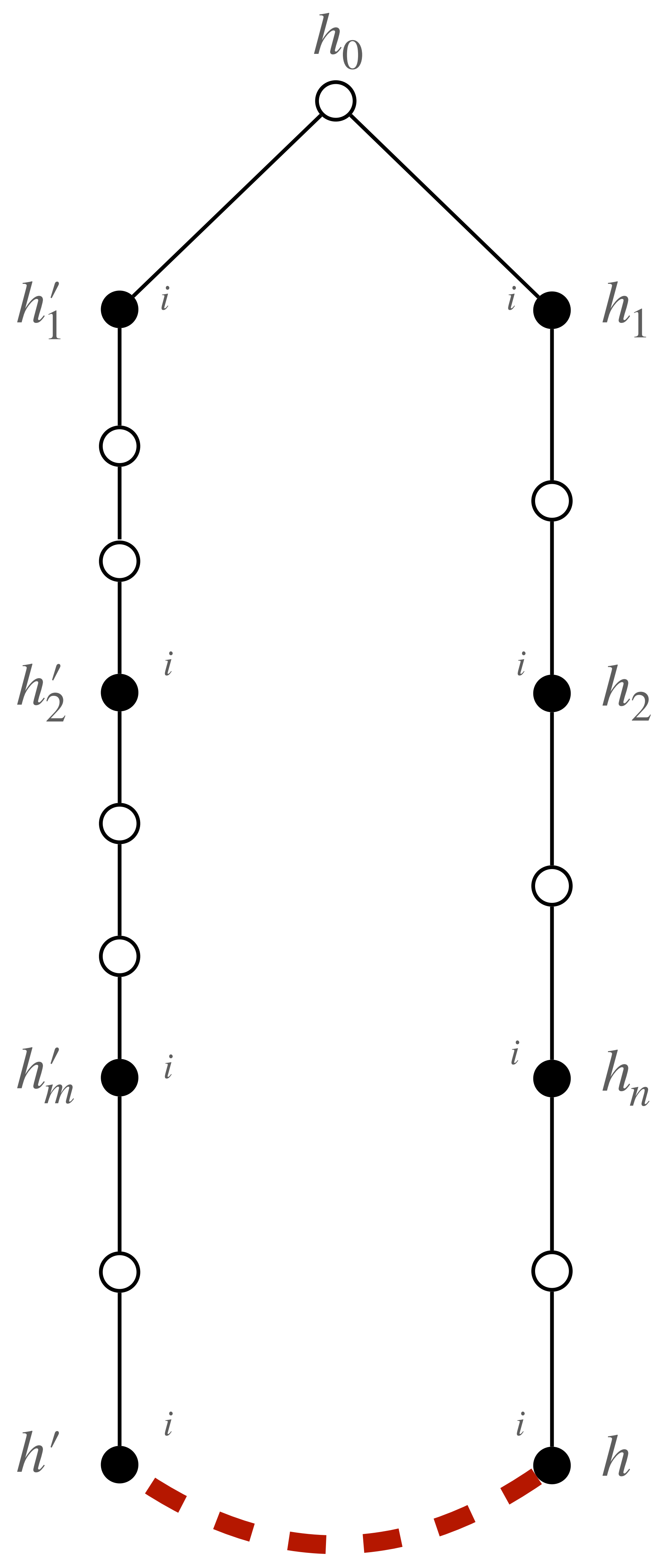
Player i has **perfect recall** in an imperfect information game G if for any two nodes h, h' that are in the same information set for player i, for any path $h_0, a_0, h_1, a_1, \ldots, h_n, h$ from the root of the game to h, and for any path $h_0, a'_0, h'_1, a'_1, \ldots, h'_m, h'$ from the root of the game to h', it must be the case that:



Perfect Recall

Definition:
Player i has **perfect recall** in an imperfect information game G if for any two nodes h, h' that are in the same information set for player i, for any path $h_0, a_0, h_1, a_1, \ldots, h_n, h$ from the root of the game to h, and for any path $h_0, a_0', h_1', a_1', \ldots, h_m', h'$ from the root of the game to h', it must be the case that:

1. $n = m$, and

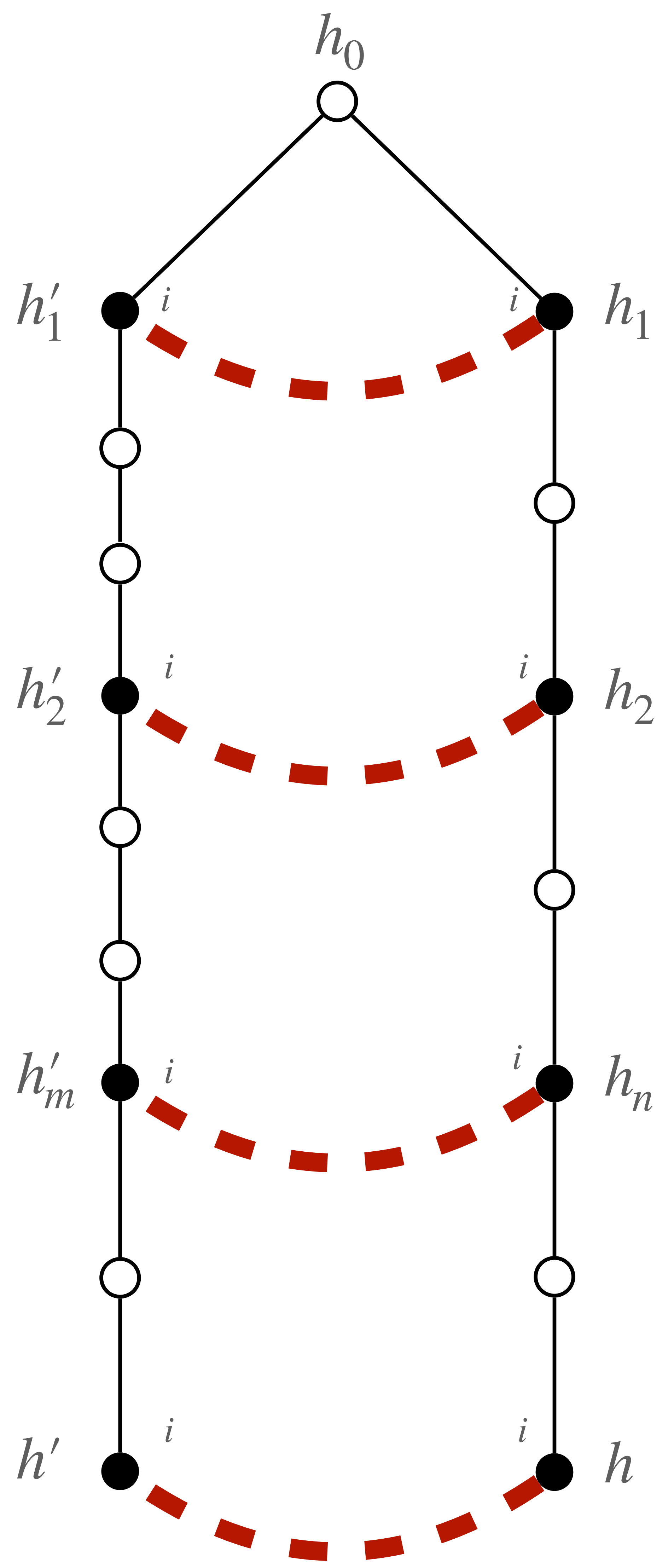


Perfect Recall

Definition:
Player i has **perfect recall** in an imperfect information game G if for any two nodes h, h' that are in the same information set for player i, for any path $h_0, a_0, h_1, a_1, \ldots, h_n, h$ from the root of the game to h, and for any path $h_0, a'_0, h'_1, a'_1, \ldots, h'_m, h'$ from the root of the game to h', it must be the case that:

1. $n = m$, and

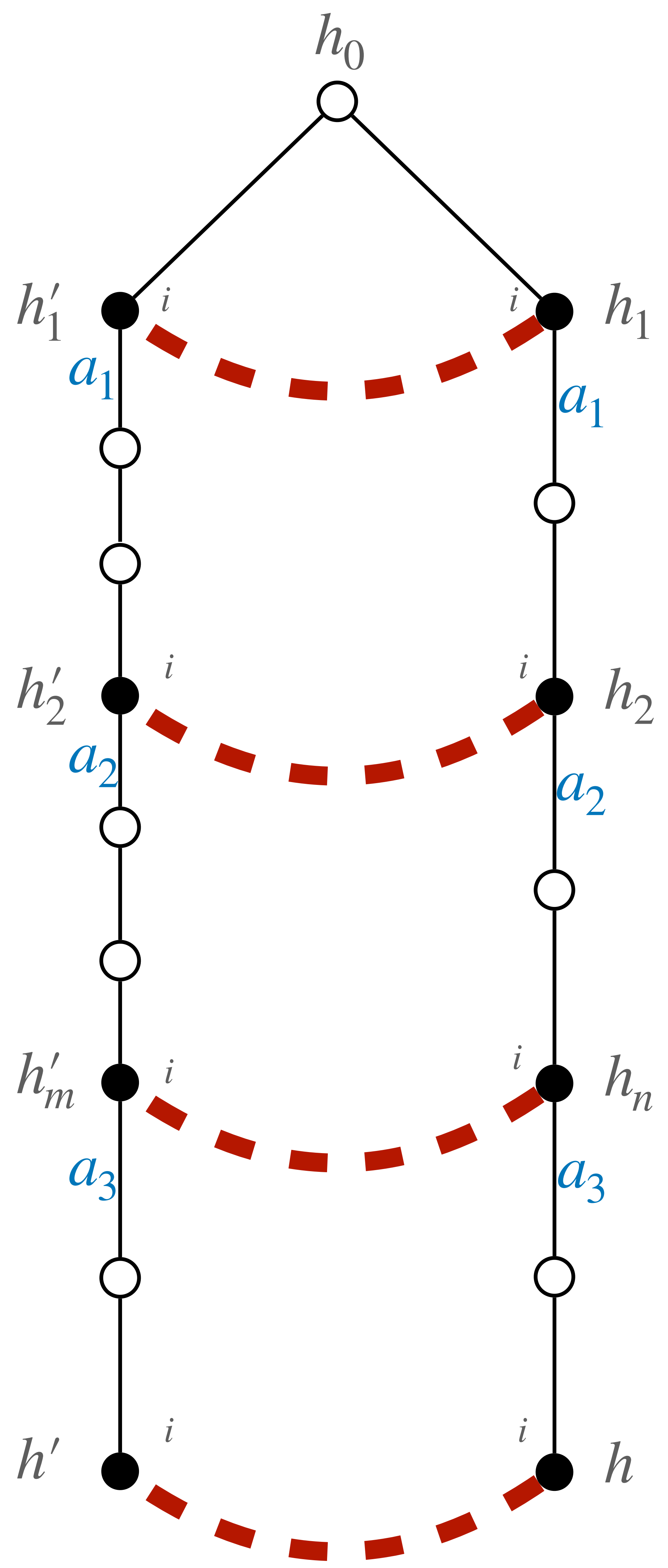
2. for all $0 \leq j \leq n$, if $\rho(h_j) = 1$, then h_j and h'_j are in the **same information set**, and



Perfect Recall

Definition:
Player i has **perfect recall** in an imperfect information game G if for any two nodes h, h' that are in the same information set for player i, for any path $h_0, a_0, h_1, a_1, \ldots, h_n, h$ from the root of the game to h, and for any path $h_0, a'_0, h'_1, a'_1, \ldots, h'_m, h'$ from the root of the game to h', it must be the case that:

1. $n = m$, and
2. for all $0 \leq j \leq n$, if $\rho(h_j) = 1$, then h_j and h'_j are in the **same information set**, and
3. for all $0 \leq j \leq n$, if $\rho(h_j) = 1$, then $a_j = a'_j$.

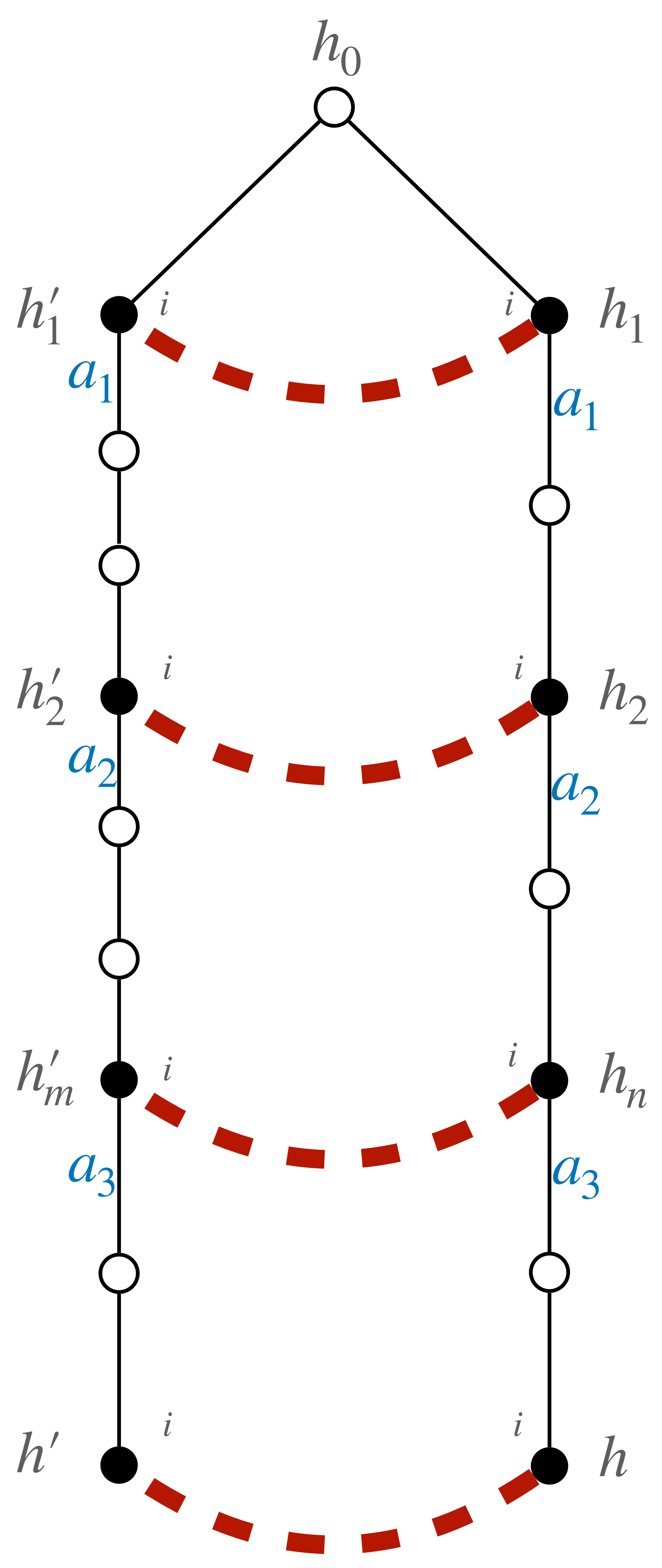


Perfect Recall

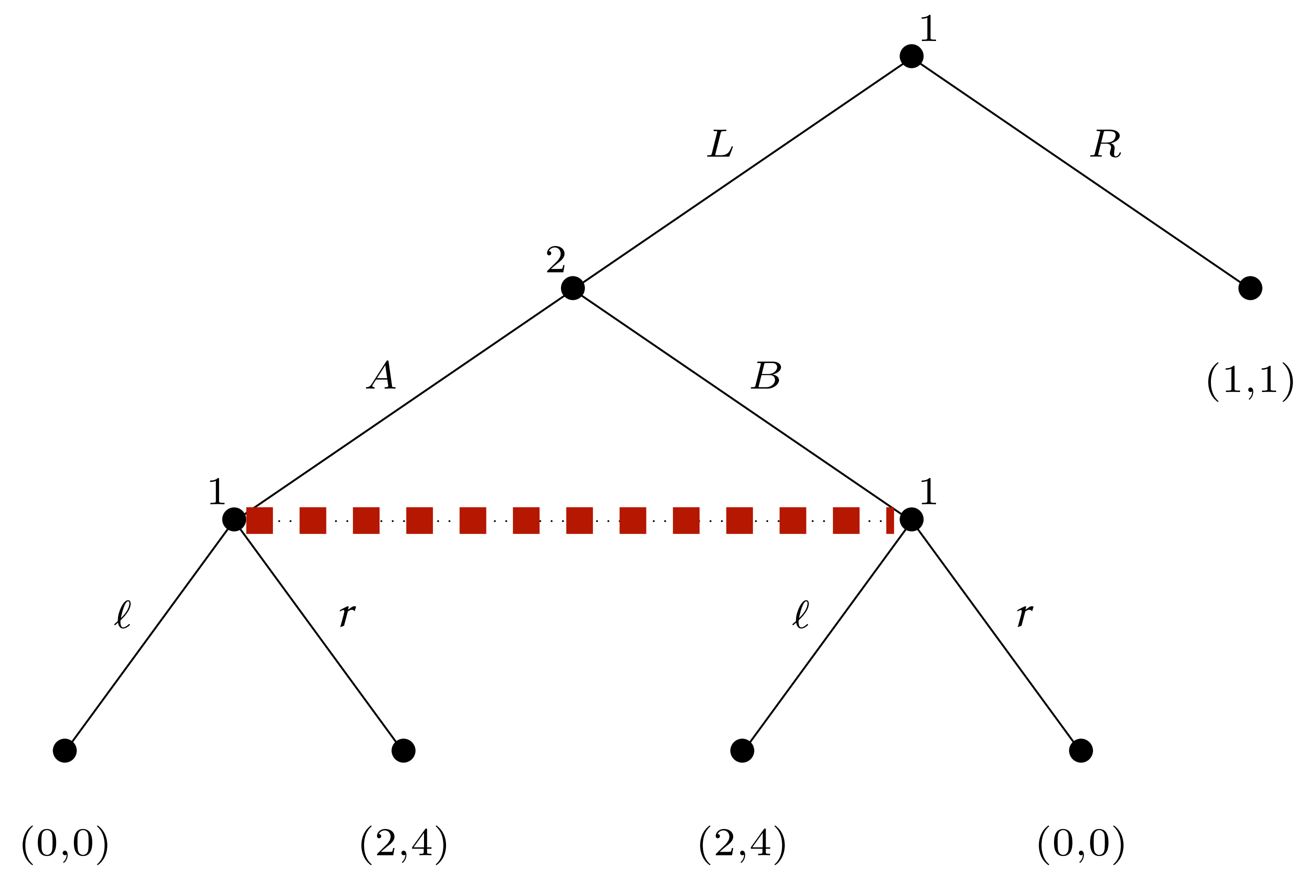
Definition:
Player i has **perfect recall** in an imperfect information game G if for any two nodes h, h' that are in the same information set for player i, for any path $h_0, a_0, h_1, a_1, \ldots, h_n, h$ from the root of the game to h, and for any path $h_0, a_0', h_1', a_1', \ldots, h_m', h'$ from the root of the game to h', it must be the case that:

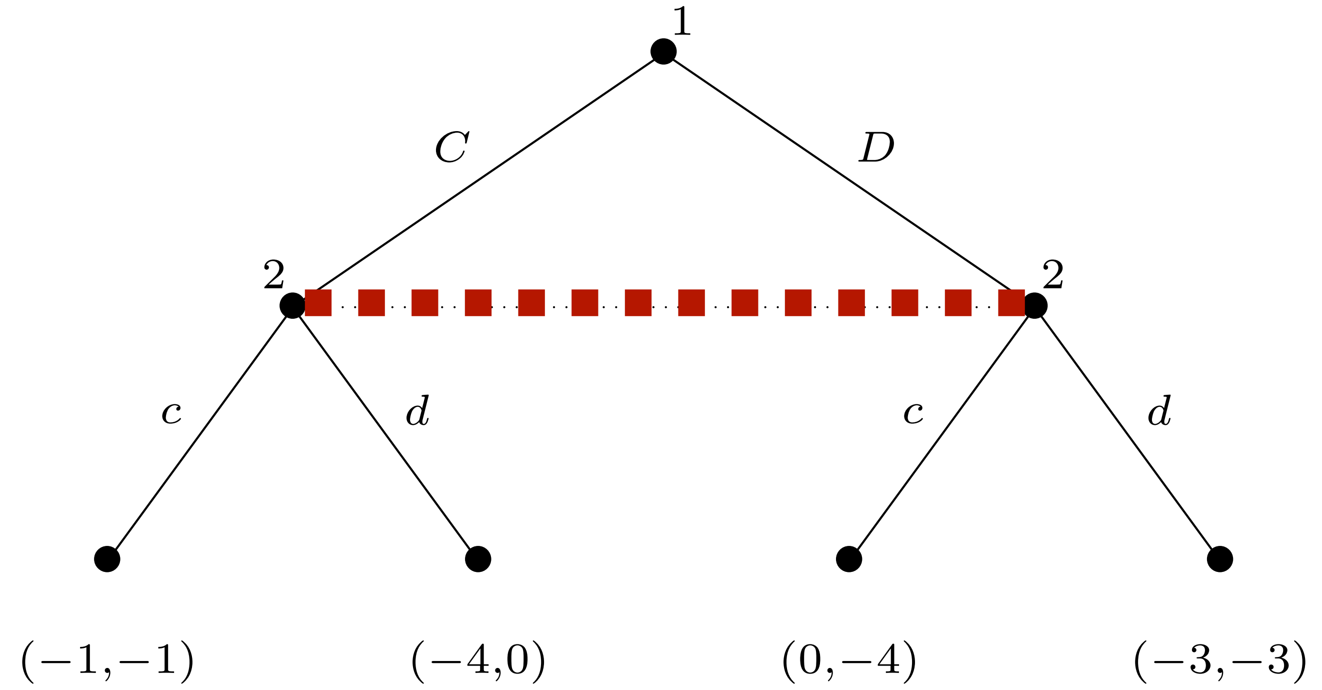
1. $n = m$, and
2. for all $0 \leq j \leq n$, if $\rho(h_j) = 1$, then h_j and h_j' are in the **same information set**, and
3. for all $0 \leq j \leq n$, if $\rho(h_j) = 1$, then $a_j = a_j'$.

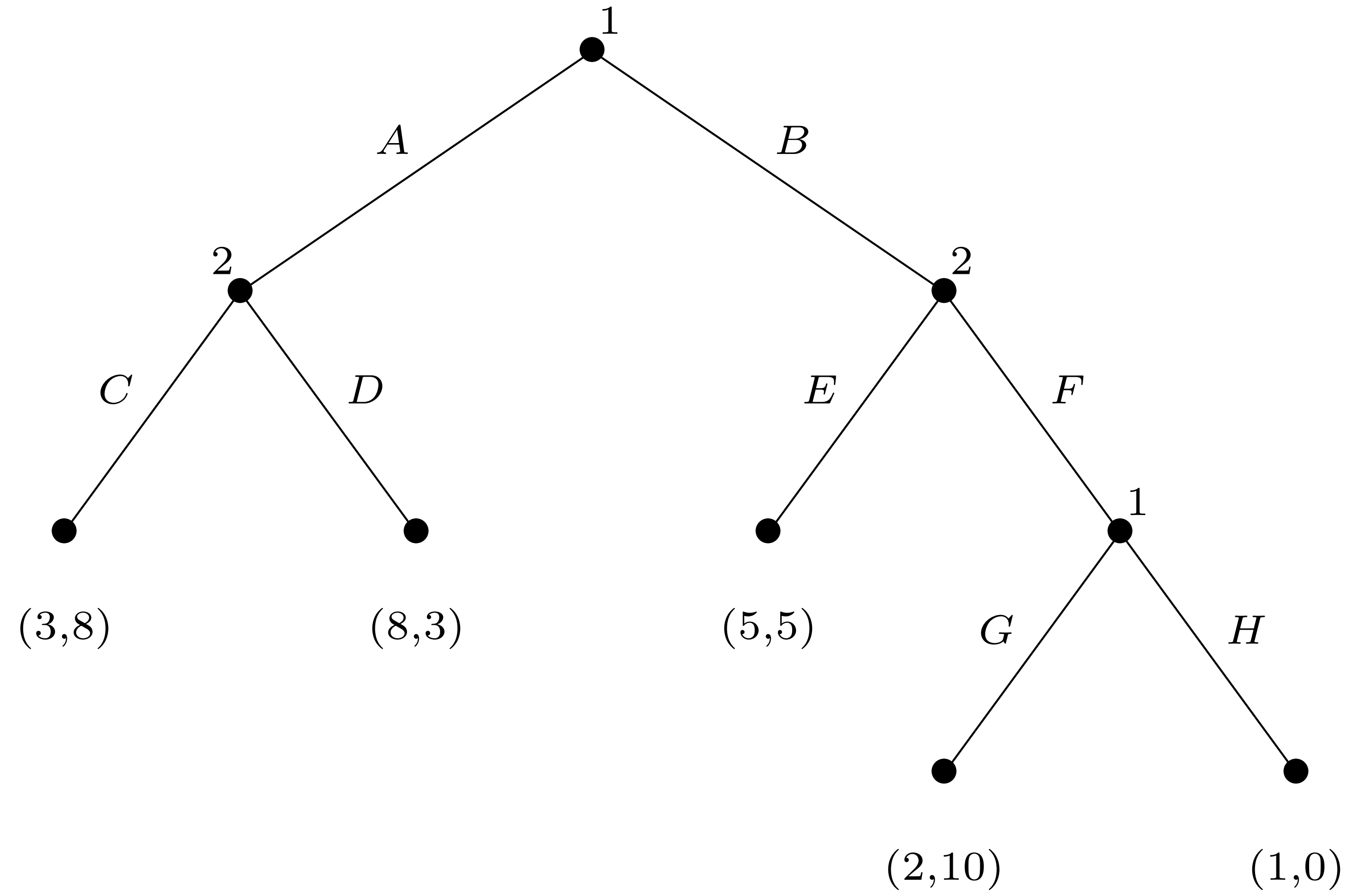
G is a **game of perfect recall** if every player has perfect recall in G.



Perfect Recall Examples

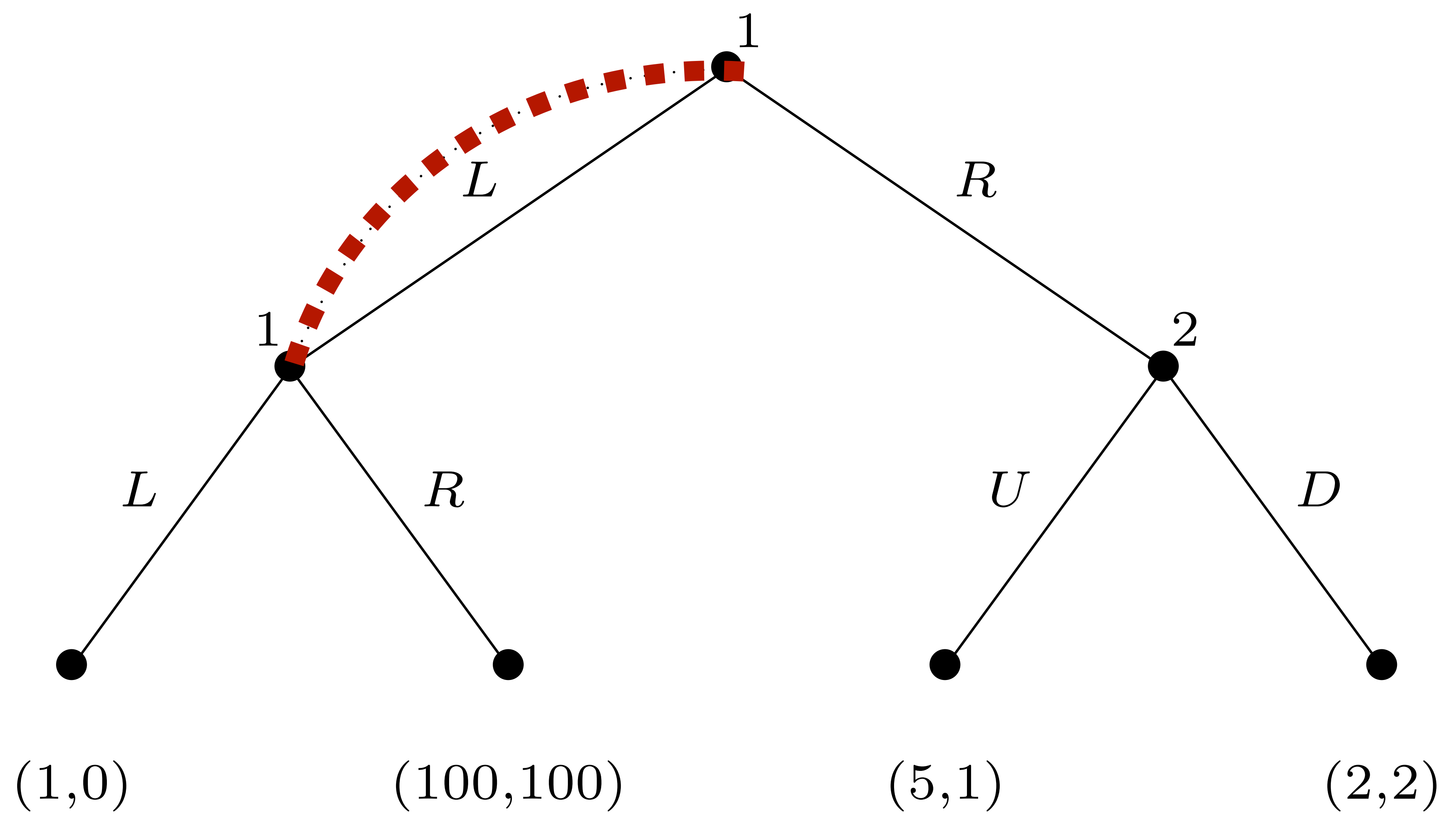






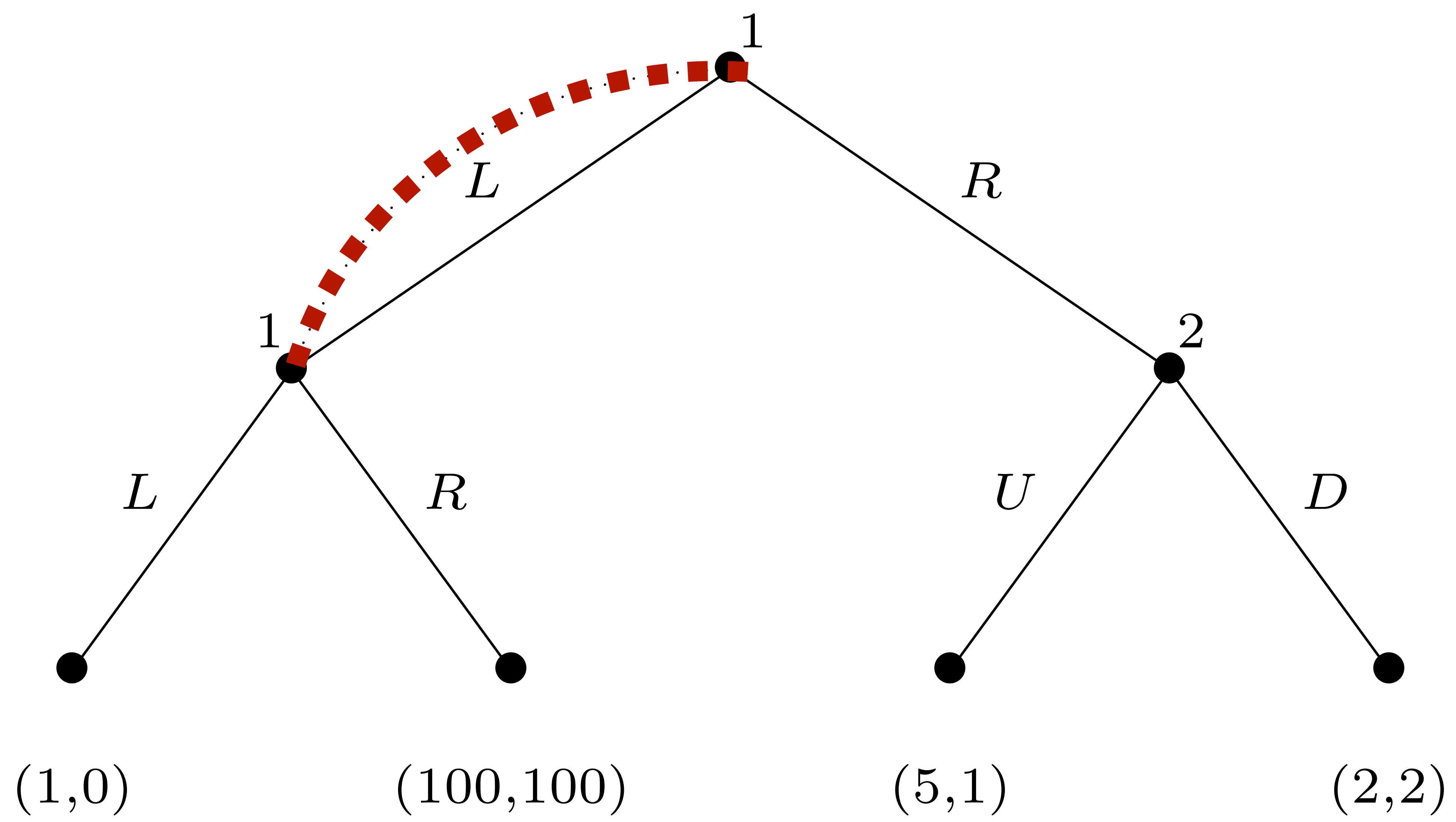
Question: Which of the above games is a game of perfect recall?

Imperfect Recall Example



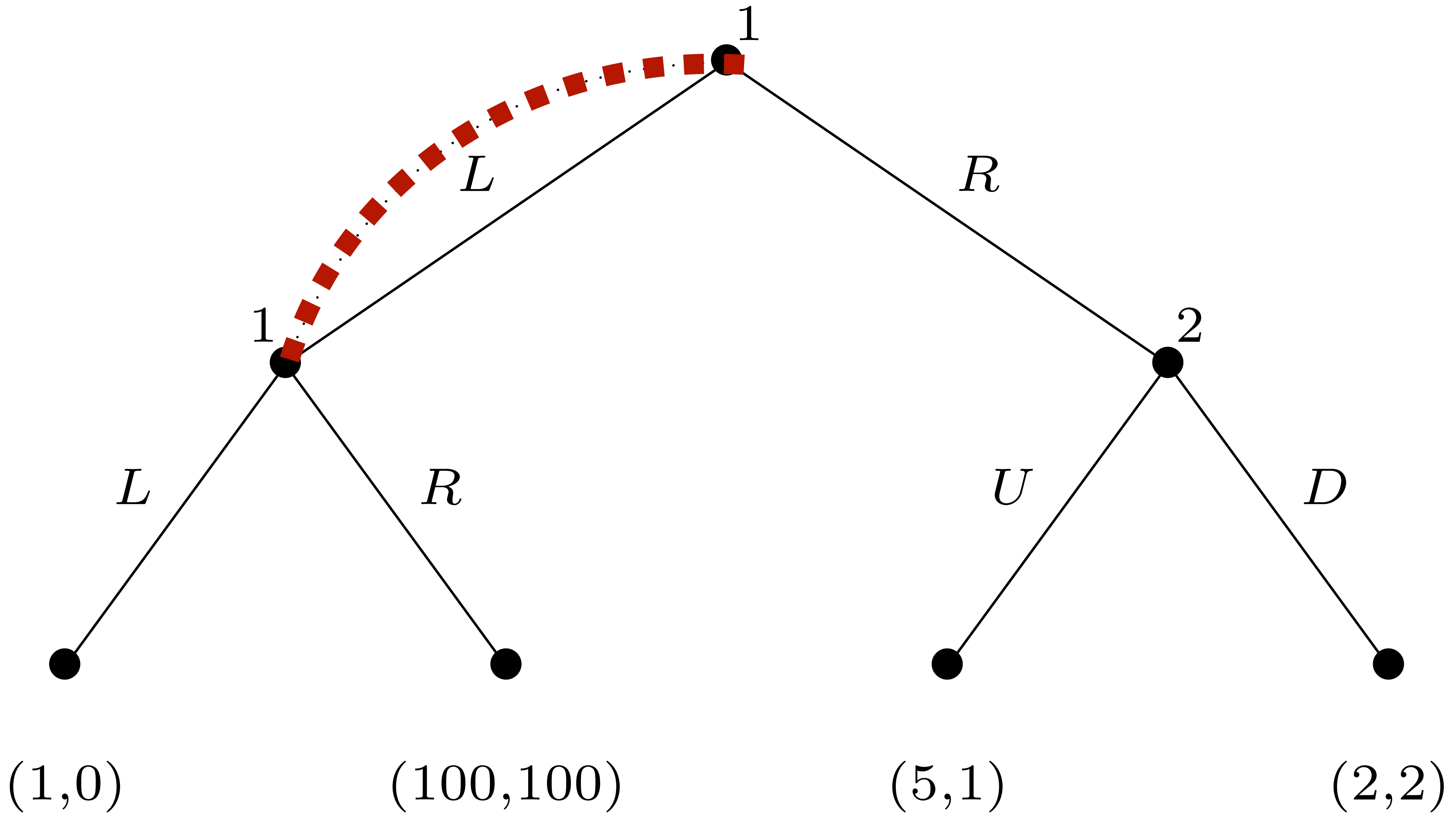
Imperfect Recall Example

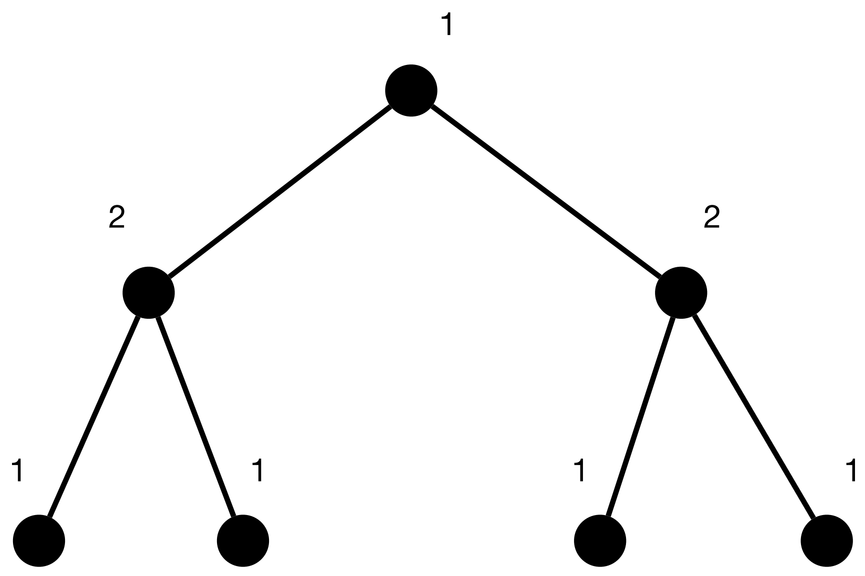
- Player 1 **doesn't remember** whether they have played *L* before or not. In this case, that is because they visit the **same information set multiple times**.



Imperfect Recall Example

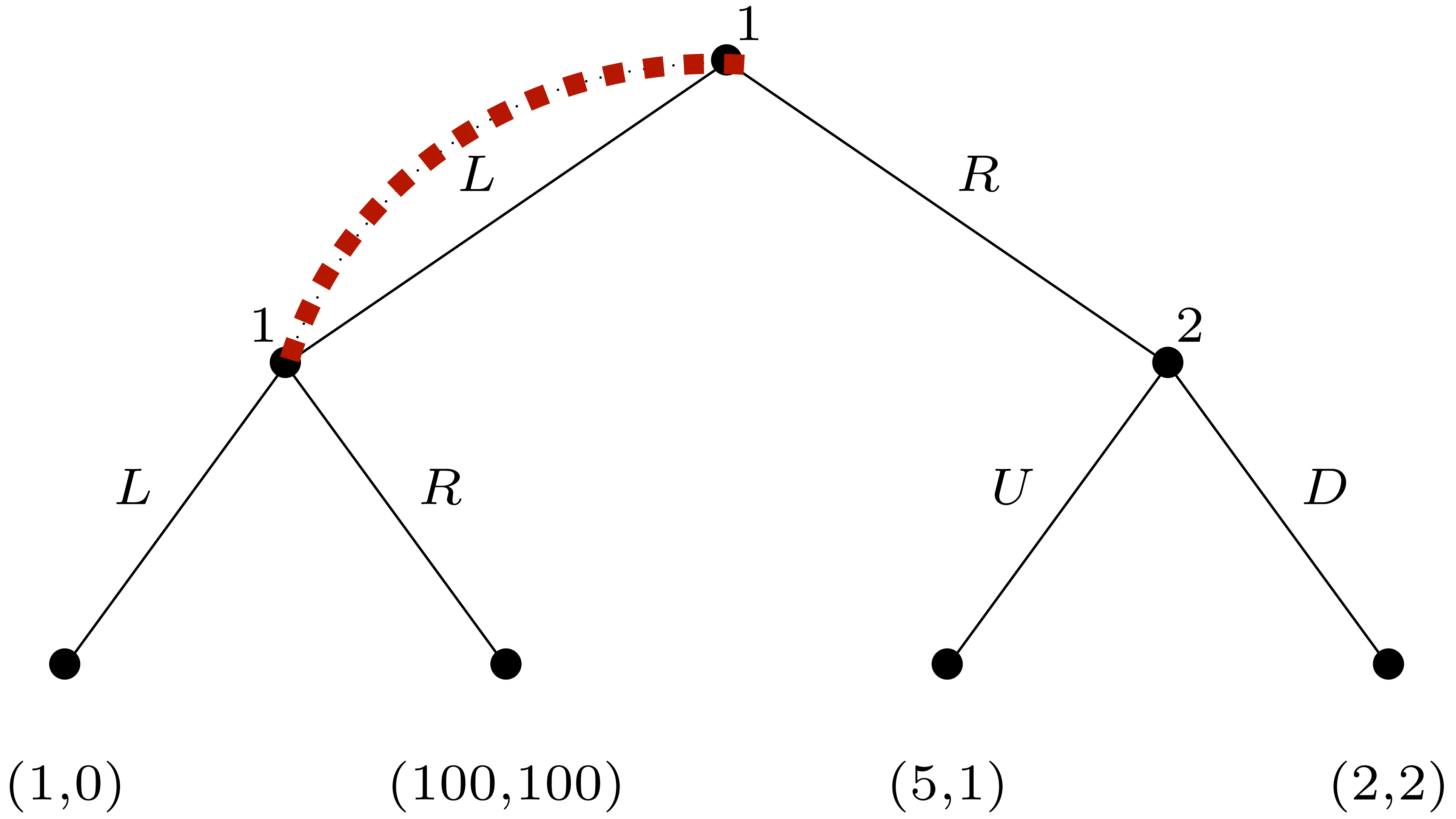
- Player 1 **doesn't remember** whether they have played *L* before or not. In this case, that is because they visit the **same information set multiple times**.

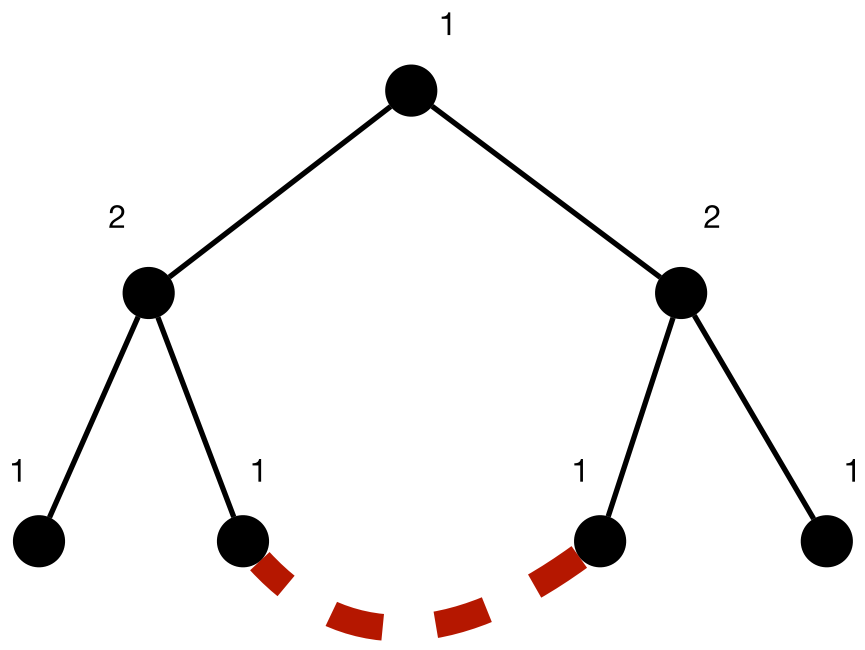




Imperfect Recall Example

- Player 1 **doesn't remember** whether they have played *L* before or not. In this case, that is because they visit the **same information set multiple times**.

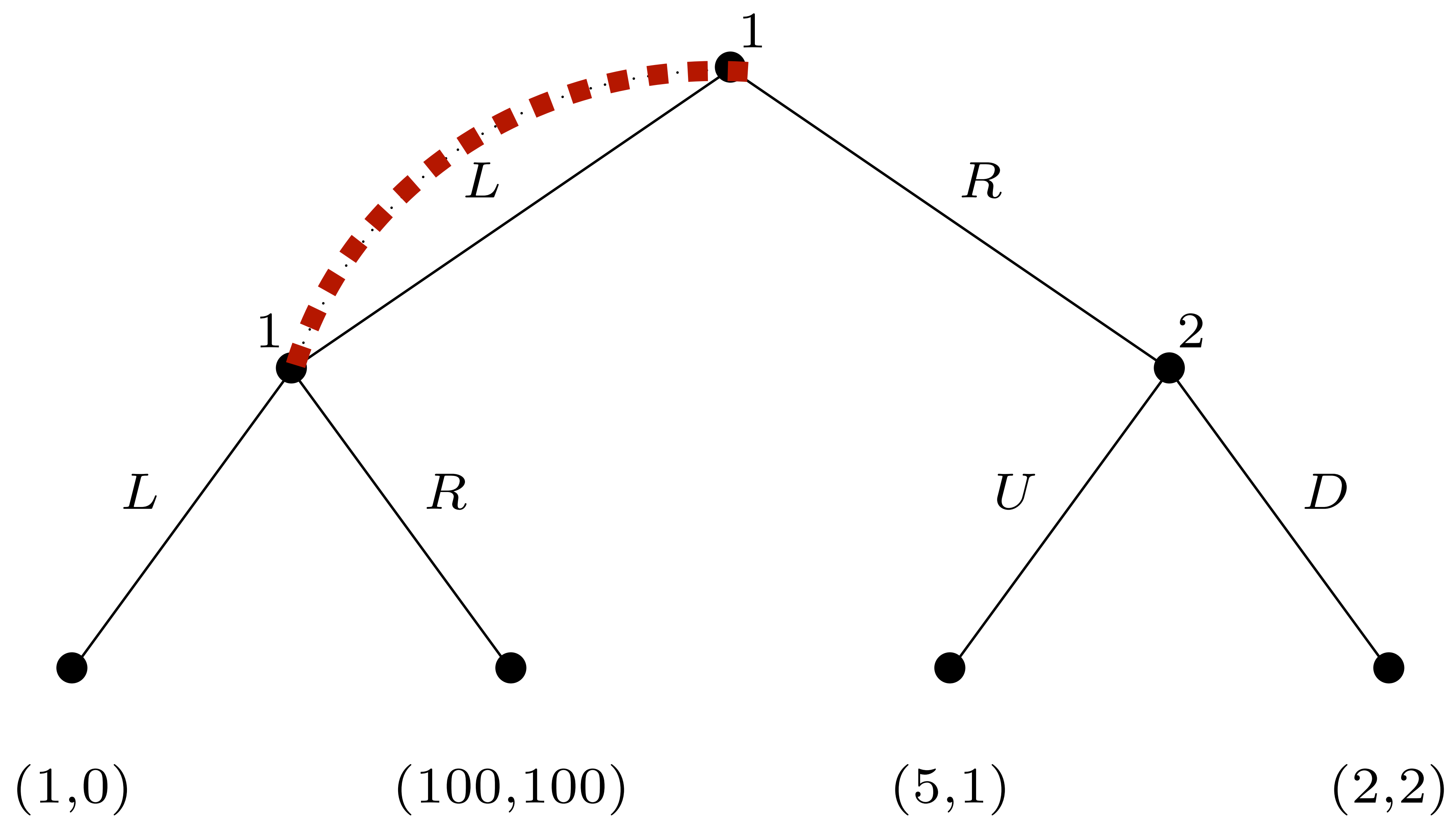




Imperfect Recall Example

- Player 1 **doesn't remember** whether they have played *L* before or not. In this case, that is because they visit the **same information set multiple times**.

- **Question:** Can you construct a **mixed strategy** equivalent to the behavioural strategy [.5:L, .5R]?

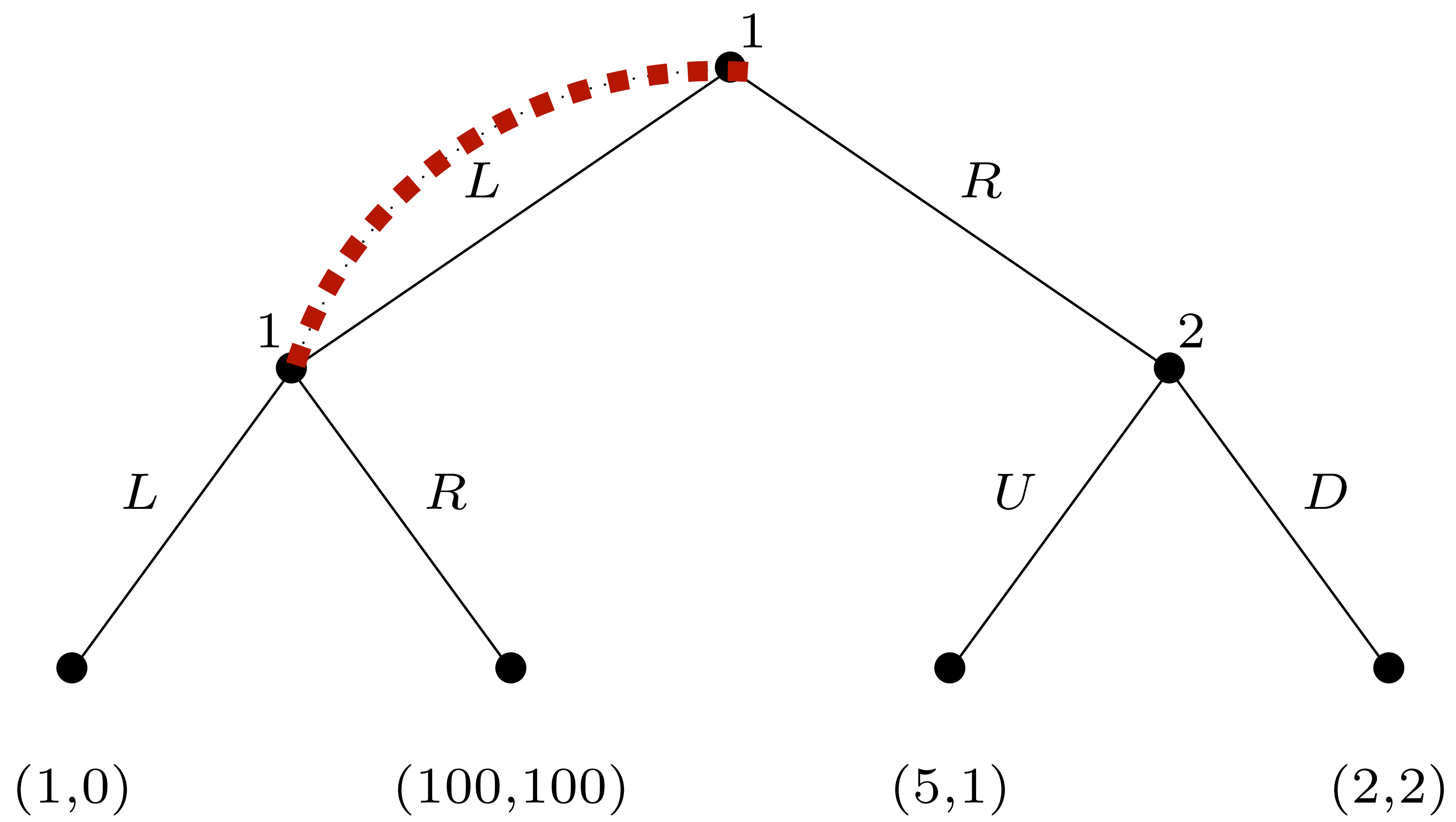


Imperfect Recall Example

- Player 1 **doesn't remember** whether they have played *L* before or not. In this case, that is because they visit the **same information set multiple times**.

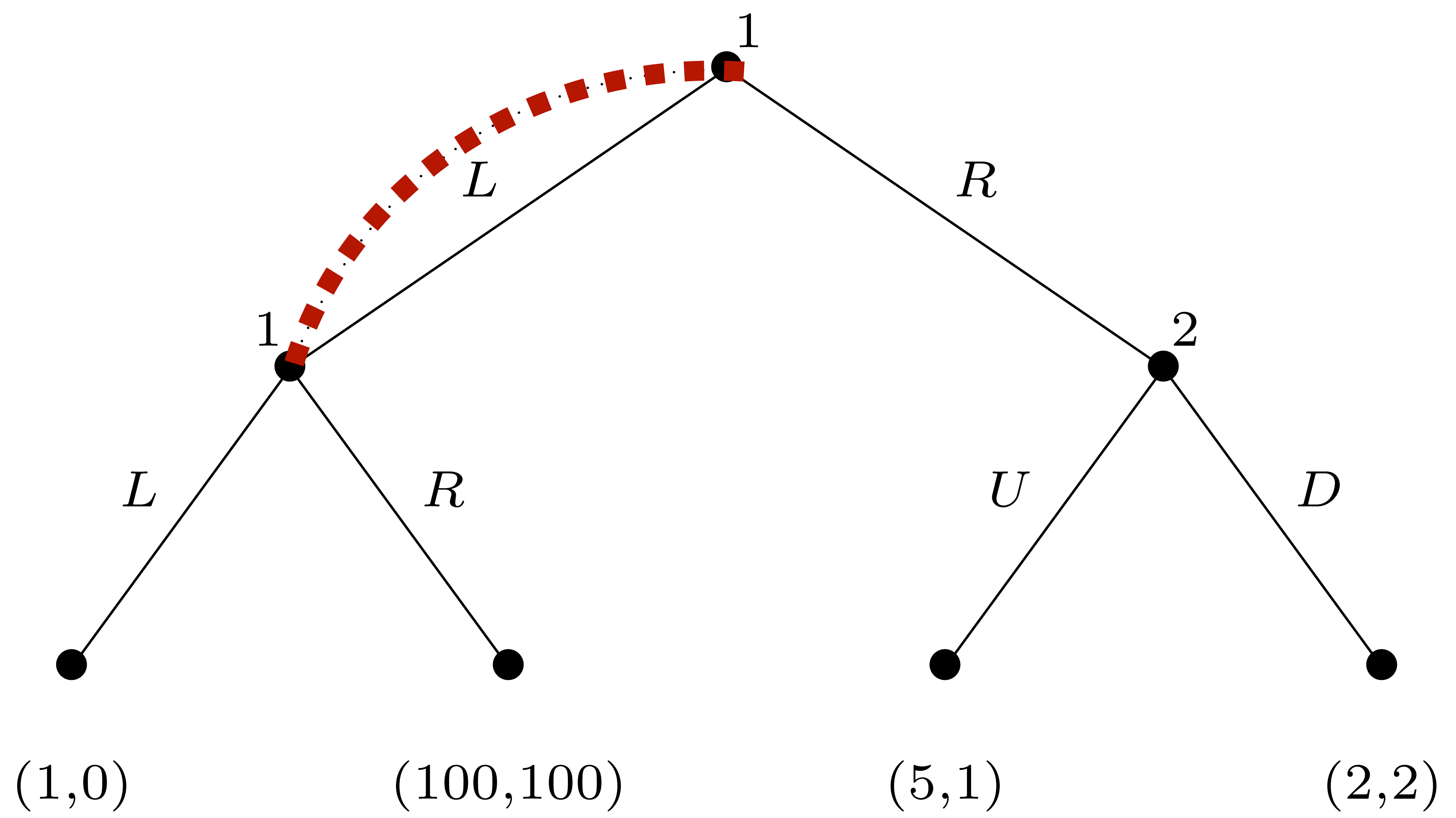
- **Question:** Can you construct a **mixed strategy** equivalent to the behavioural strategy [.5:L, .5R]?

- **Question:** Can you construct a **behavioural strategy** equivalent to the mixed strategy [.5:L, .5:R]?



Imperfect Recall Example

- Player 1 **doesn't remember** whether they have played *L* before or not. In this case, that is because they visit the **same information set multiple times**.
- **Question:** Can you construct a **mixed strategy** equivalent to the behavioural strategy [.5:L, .5R]?
- **Question:** Can you construct a **behavioural strategy** equivalent to the mixed strategy [.5:L, .5:R]?
- **Question:** What is the **mixed strategy equilibrium** in this game?



Imperfect Recall Example

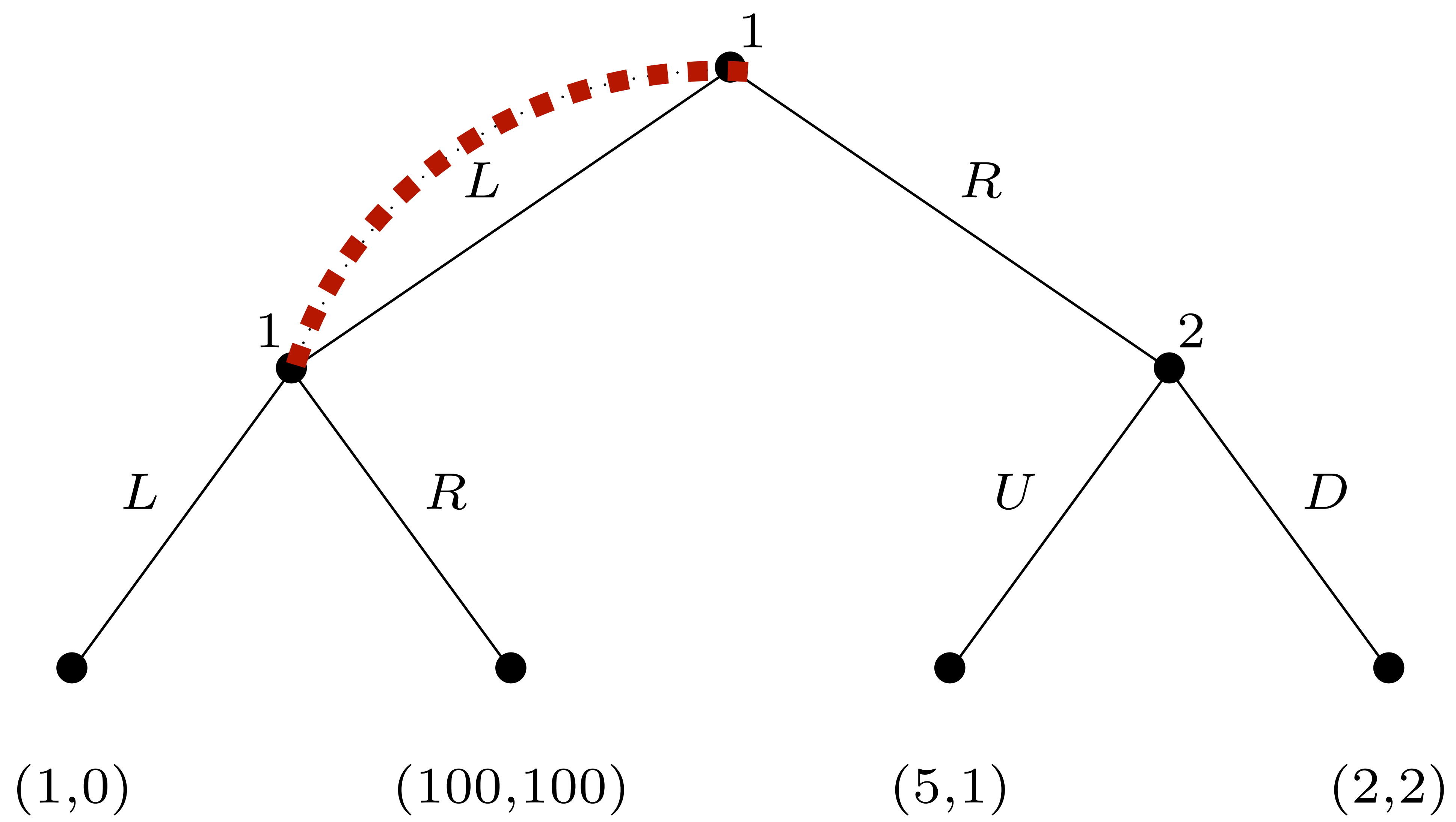
- Player 1 **doesn't remember** whether they have played *L* before or not. In this case, that is because they visit the **same information set multiple times**.

- **Question:** Can you construct a **mixed strategy** equivalent to the behavioural strategy [.5:L, .5R]?

- **Question:** Can you construct a **behavioural strategy** equivalent to the mixed strategy [.5:L, .5:R]?

- **Question:** What is the **mixed strategy equilibrium** in this game?

- **Question:** What is an **equilibrium in behavioural strategies** in this game?



Imperfect Recall Applications

Question: When is it **useful** to model a scenario as a game of **imperfect recall**?

Imperfect Recall Applications

Question: When is it **useful** to model a scenario as a game of **imperfect recall**?

1. When the **actual agents** being modelled may **forget** previous history
    - Including cases where the agents strategies really are executed by **proxies**

Imperfect Recall Applications

Question: When is it **useful** to model a scenario as a game of **imperfect recall**?

1. When the **actual agents** being modelled may **forget** previous history
   - Including cases where the agents strategies really are executed by **proxies**
2. As an **approximation technique**
   - E.g., **poker**: The exact cards that have been played to this point may not matter as much as some coarse grouping of which cards have been played
   - Grouping the cards into equivalence classes is a **lossy** approximation

Kuhn's Theorem

Kuhn's Theorem

Theorem: [Kuhn, 1953]
In a game of perfect recall, any **mixed strategy** of a given agent can be **replaced by an equivalent behavioural strategy**, and any **behavioural strategy** can be **replaced by an equivalent mixed strategy**.

Kuhn's Theorem

Theorem: [Kuhn, 1953]
In a game of perfect recall, any **mixed strategy** of a given agent can be **replaced by an equivalent behavioural strategy**, and any **behavioural strategy** can be **replaced by an equivalent mixed strategy**.

- Here, two strategies are **equivalent** when they induce the same probabilities on outcomes, for any fixed strategy profile (mixed or behavioural) of the other agents.

Kuhn's Theorem

Theorem: [Kuhn, 1953]
In a game of perfect recall, any **mixed strategy** of a given agent can be **replaced by an equivalent behavioural strategy**, and any **behavioural strategy** can be **replaced by an equivalent mixed strategy**.

- Here, two strategies are **equivalent** when they induce the same probabilities on outcomes, for any fixed strategy profile (mixed or behavioural) of the other agents.

Corollary:
Restricting attention to behavioural strategies does not change the set of Nash equilibria in a game of perfect recall. (**why**?)

Computational Issues

Computational Issues

- **Question:** Can we use **backward induction** to find an equilibrium in an imperfect information extensive form game?

Computational Issues

- **Question:** Can we use **backward induction** to find an equilibrium in an imperfect information extensive form game?
- We can just use the **induced normal form** to find the equilibrium of any imperfect information game

Computational Issues

- **Question:** Can we use **backward induction** to find an equilibrium in an imperfect information extensive form game?

- We can just use the **induced normal form** to find the equilibrium of any imperfect information game

    - But the induced normal form is **exponentially larger** than the extensive form

Computational Issues

- **Question:** Can we use **backward induction** to find an equilibrium in an imperfect information extensive form game?
- We can just use the **induced normal form** to find the equilibrium of any imperfect information game
    - But the induced normal form is **exponentially larger** than the extensive form
- Can use the **sequence form** [S&LB §5.2.3] in games of **perfect recall**:

Computational Issues

- **Question:** Can we use **backward induction** to find an equilibrium in an imperfect information extensive form game?
- We can just use the **induced normal form** to find the equilibrium of any imperfect information game
  - But the induced normal form is **exponentially larger** than the extensive form
- Can use the **sequence form** [S&LB §5.2.3] in games of **perfect recall**:
  - **Zero-sum games:** **polynomial** in size of extensive form
    (i.e., exponentially faster than LP formulation on normal form)

Computational Issues

- **Question:** Can we use **backward induction** to find an equilibrium in an imperfect information extensive form game?
- We can just use the **induced normal form** to find the equilibrium of any imperfect information game
    - But the induced normal form is **exponentially larger** than the extensive form
- Can use the **sequence form** [S&LB §5.2.3] in games of **perfect recall**:
    - **Zero-sum games:** **polynomial** in size of extensive form (i.e., exponentially faster than LP formulation on normal form)
    - **General-sum games:** **exponential** in size of extensive form (i.e., exponentially faster than converting to normal form)

Summary

Summary

- **Imperfect information extensive form games** are a model of games with sequential actions, some of which may be **hidden**
    - Histories are partitioned into **information sets**
    - Player **cannot distinguish** between histories in the same information set

Summary

- **Imperfect information extensive form games** are a model of games with sequential actions, some of which may be **hidden**
    - Histories are partitioned into **information sets**
    - Player **cannot distinguish** between histories in the same information set
- **Pure strategies** map each information set to an action
    - **Mixed strategies** are distributions over pure strategies
    - **Behavioural strategies** map each information set to a distribution over actions

Summary

- **Imperfect information extensive form games** are a model of games with sequential actions, some of which may be **hidden**
  - Histories are partitioned into **information sets**
  - Player **cannot distinguish** between histories in the same information set
- **Pure strategies** map each information set to an action
  - **Mixed strategies** are distributions over pure strategies
  - **Behavioural strategies** map each information set to a distribution over actions
  - In games of perfect recall, mixed strategies and behavioural strategies are **interchangeable**

Summary

- **Imperfect information extensive form games** are a model of games with sequential actions, some of which may be **hidden**
    - Histories are partitioned into **information sets**
    - Player **cannot distinguish** between histories in the same information set
- **Pure strategies** map each information set to an action
    - **Mixed strategies** are distributions over pure strategies
    - **Behavioural strategies** map each information set to a distribution over actions
    - In games of perfect recall, mixed strategies and behavioural strategies are **interchangeable**
- A player has **perfect recall** if they **never forget** anything they knew about actions so far