# Game Theory for Sequential Interactions

CMPUT 366: Intelligent Systems

S&LB §5.0-5.2.2

# Lecture Outline

1. Recap & Logistics
2. Mixed Strategies
3. Perfect Information Games
4. Backward Induction

# Logistics

- **Assignment 4** is due **Friday April 15** at 11:59pm
- **USRIs** are now available for this course:
  - You should have gotten an email
  - Can also access at: https://p20.courseval.net/etw/ets/et.asp?nxappid=UA2&nxmid=start
  - Survey is available until **Friday April 8** at 11:59pm
- **Assignment 3** marks should be available by the end of the week
- **Solutions** to midterm and assignment 3 are available on eClass

# Recap: Game Theory

- Game theory studies the **interactions of rational agents**
    - Canonical representation is the **normal form game**
- Game theory uses **solution concepts** rather than optimal behaviour
    - "Optimal behaviour" is not clear-cut in multiagent settings
    - **Pareto optimal:** no agent can be made better off without making some other agent worse off
    - **Nash equilibrium:** no agent regrets their strategy given the choice of the other agents' strategies

| | Ballet | Soccer |
|---|---|---|
| Ballet | 2, 1 | 0, 0 |
| Soccer | 0, 0 | 1, 2 |

# Nash Equilibria of Examples

The only equilibrium of Prisoner's Dilemma is also the *only* outcome that is Pareto-dominated!

| | Coop. | Defect |
|---|---|---|
| Coop. | -1,-1 | -5,0 |
| Defect | 0,-5 | -3,-3 |

| | Left | Right |
|---|---|---|
| Left | 1 | -1 |
| Right | -1 | 1 |

| | Ballet | Soccer |
|---|---|---|
| Ballet | 2, 1 | 0, 0 |
| Soccer | 0, 0 | 1, 2 |

| | Heads | Tails |
|---|---|---|
| Heads | 1,-1 | -1,1 |
| Tails | -1,1 | 1,-1 |

# Mixed Strategies

**Definitions:**

- A **strategy** $s_i$ for agent $i$ is any probability distribution over the set $A_i$, where each action $a_i$ is played with probability $s_i(a_i)$.
  - **Pure strategy:** only a single action is played
  - **Mixed strategy:** randomize over multiple actions
- Set of $i$'s strategies: $S_i \doteq \Delta(A_i)$ ← $\Delta(X)$ = "set of distributions over elements of $X$"
- Set of **strategy profiles:** $S = S_1 \times S_2 \times \cdots \times S_n$
- **Utility** of a mixed strategy profile:

$$u_i(s) \doteq \mathbb{E}[u(A) \mid A_i \sim s_i] = \sum_{a \in A} \underbrace{\boxed{u_i(a)}}_{\text{Utility of action profile } a} \underbrace{\boxed{\prod_{j \in N} s_j(a_j)}}_{\text{Probability of profile } a \text{ given all agents play according to } s}$$

# Best Response and Nash Equilibrium

**Definition:**
The set of $i$'s **best responses** to a strategy profile $s \in S$ is

$$BR_i(s_{-i}) \doteq \{s_i^* \in S_i \mid u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i}) \ \ \forall s_i \in S_i\}$$

**Definition:**
A strategy profile $s \in S$ is a **Nash equilibrium** iff

$$\forall i \in N \quad s_i \in BR_{-i}(s_{-i})$$

- When at least one $s_i$ is mixed, $s$ is a **mixed strategy Nash equilibrium**

# Nash's Theorem

**Theorem:** [Nash 1951]
Every game with a finite number of players and action profiles has at least one Nash equilibrium.

- **Pure strategy** equilibria are *not* guaranteed to exist

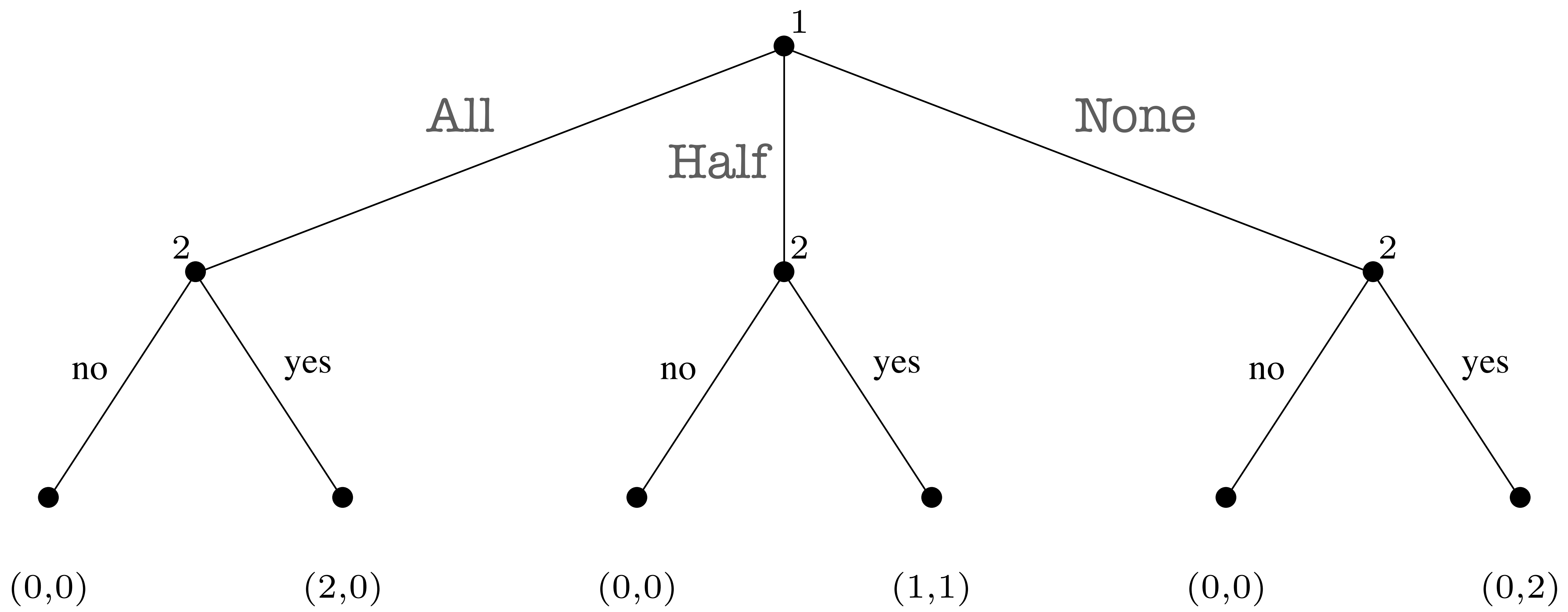
# Interpreting Mixed Strategy Nash Equilibrium

What does it even mean to say that agents are playing a mixed strategy Nash equilibrium?

- They truly are **sampling a distribution** in their heads, perhaps to **confuse** their opponents (e.g., soccer, other zero-sum games)
- The distribution represents the **other agents' uncertainty** about what the agent will do
- The distribution is the **empirical frequency** of actions in repeated play
- The distribution is the frequency of a pure strategy in a **population** of pure strategies (i.e., every individual plays a pure strategy)

# Extensive Form Games

- Normal form games don't have any notion of **sequence**: all actions happen **simultaneously**
- The **extensive form** is a game representation that explicitly includes temporal structure (i.e., a **game tree**)



# Perfect Information

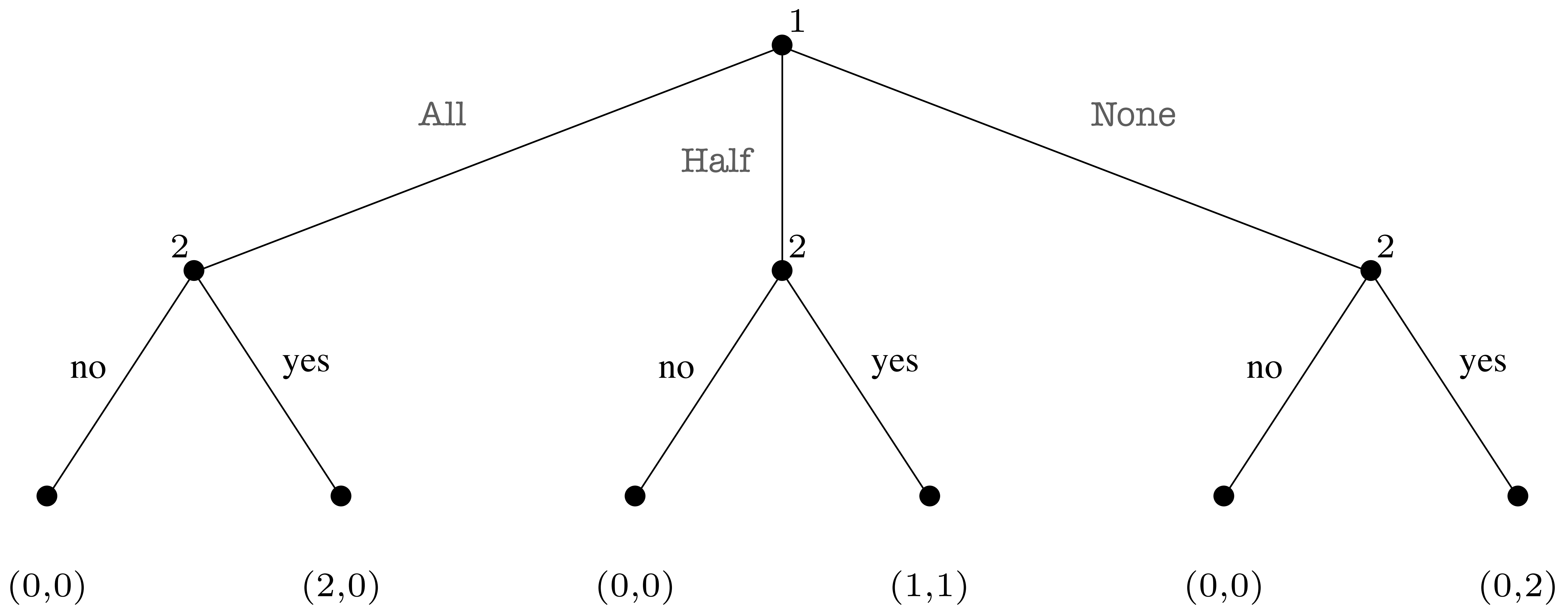
There are two kinds of extensive form game:

1. **Perfect information:** Every agent **sees all actions** of the other players (including "**Nature**")
    - e.g.: Chess, checkers, Pandemic
2. **Imperfect information:** Some actions are **hidden**
    - Players may not know exactly where they are in the tree
    - e.g.: Poker, rummy, Scrabble

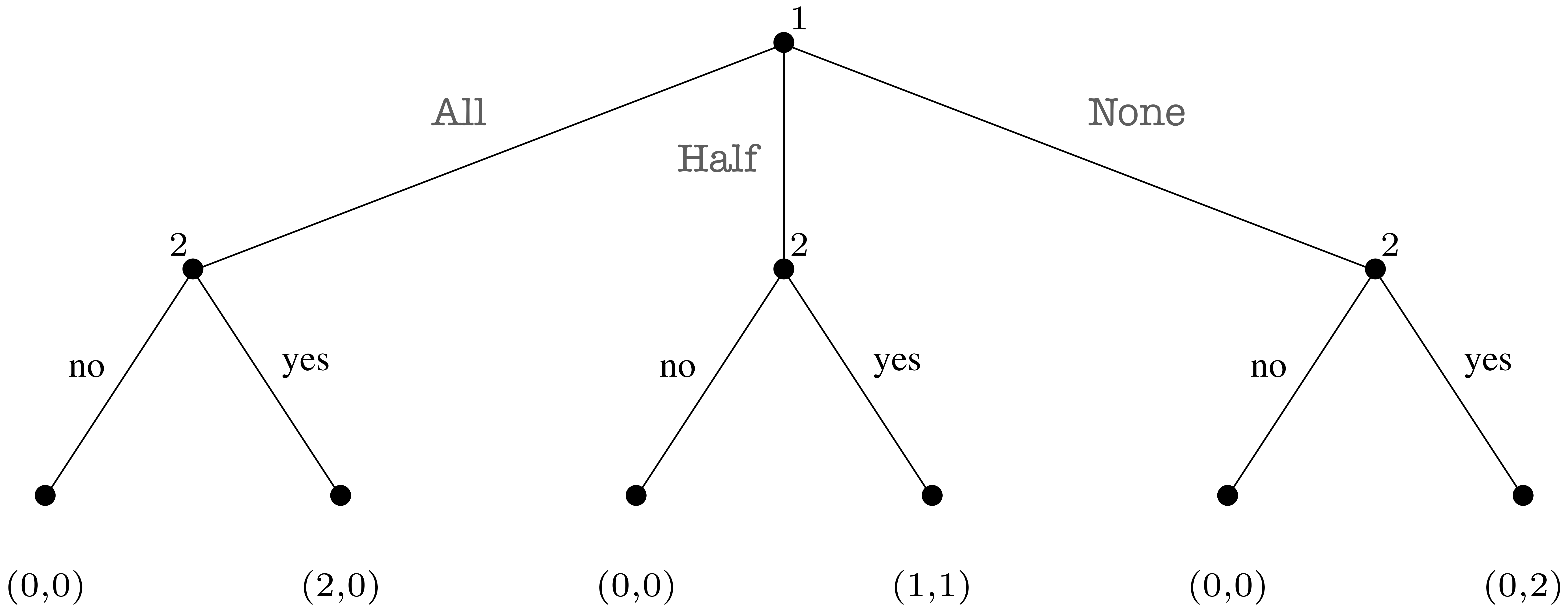
# Perfect Information Extensive Form Game

**Definition**:
A **finite perfect-information game in extensive form** is a tuple $G = (N, A, H, Z, \chi, \rho, \sigma, u)$, where

- $N$ is a set of $n$ **players**,
- $A$ is a single set of **actions**,
- $H$ is a set of nonterminal **choice nodes**,
- $Z$ is a set of **terminal nodes** (disjoint from $H$),
- $\chi : H \to 2^A$ is the **action function**,
- $\rho : H \to N$ is the **player function**,
- $\sigma : H \times A \to H \cup Z$ is the **successor function**,
- $u = (u_1, u_2, \ldots, u_n)$ is a **utility function** for each player, $u_i : Z \to \mathbb{R}$



# Fun Game:
# The Sharing Game




- Two siblings must decide how to share two $100 coins
- Sibling 1 suggests a division, then sibling 2 **accepts** or **rejects**
    - If **rejected**, nobody gets any coins.

# Pure Strategies

**Question:** What are the **pure strategies** in an extensive form game?

**Definition:**
Let $G = (N, A, H, Z, \chi, \rho, \sigma, u)$ be a perfect information game in extensive form. Then the **pure strategies of player $i$** consist of the cross product of actions available to player $i$ at each of their choice nodes, i.e.,

$$\prod_{h \in H | \rho(h) = i} \chi(h)$$

- A pure strategy associates an action with **each** choice node, even those that will **never be reached**
  - Even nodes that will never be reached as a result of the strategy itself!

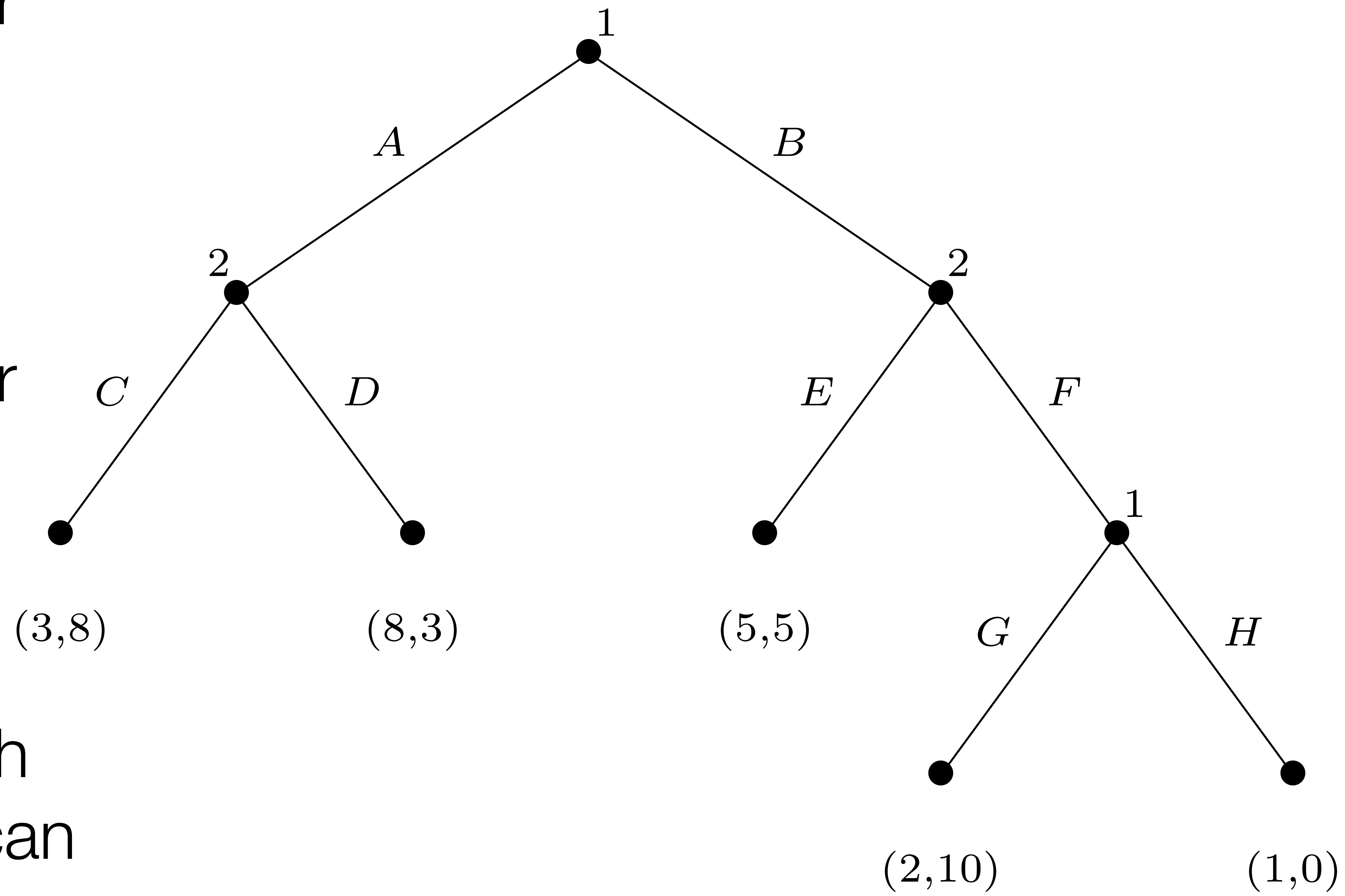
# Pure Strategies Example

**Question:** What are the **pure strategies** for **player 2**?

- $\{(C,E),(C,F),(D,E),(D,F)\}$

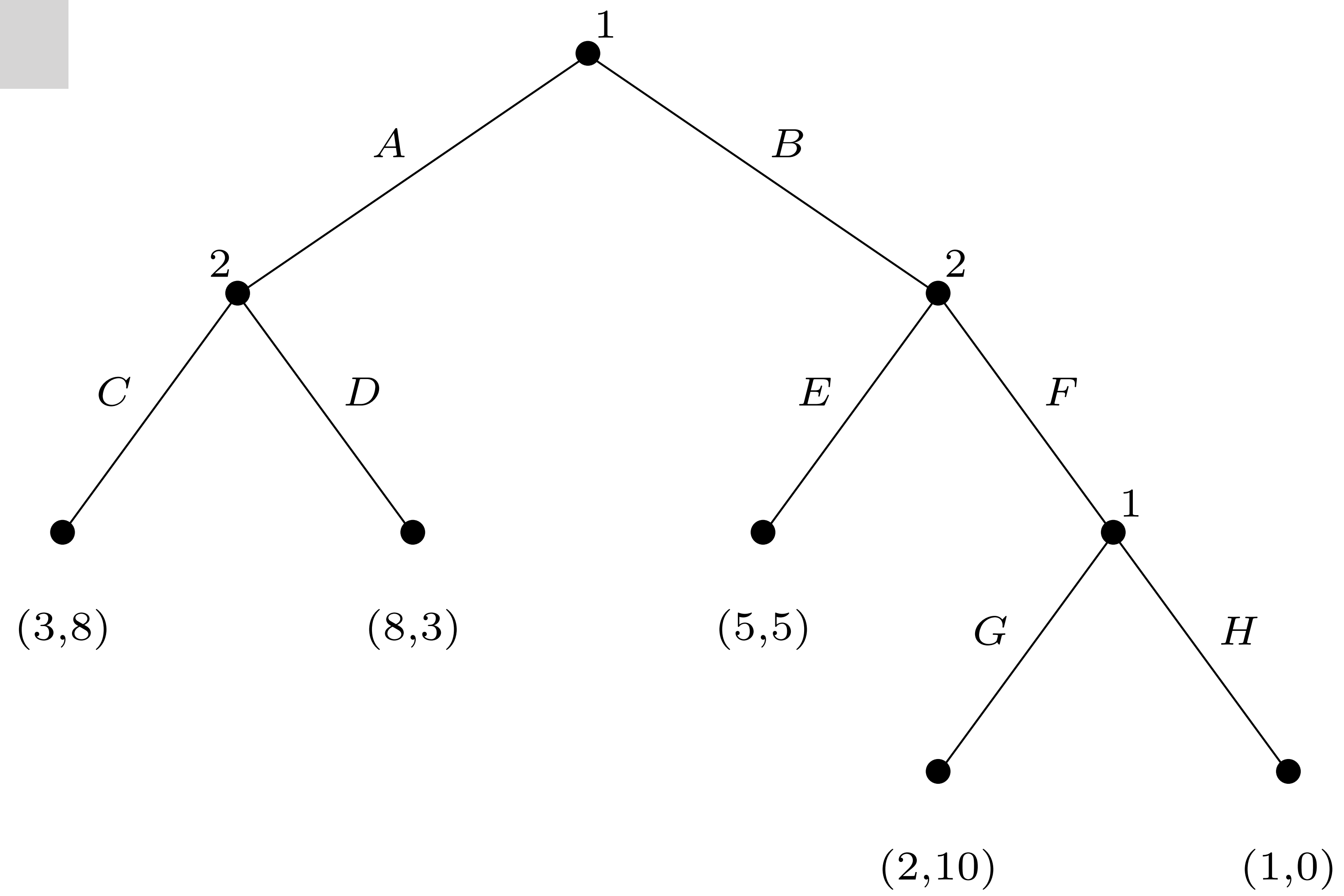
**Question:** What are the **pure strategies** for **player 1**?

- $\{(A,G),(A,H),(B,G),(B,H)\}$
- Note that these associate an action with the second choice node even when it can never be reached



# Induced Normal Form

**Question:**
Which representation is more **compact**?




| | C,E | C,F | D,E | D,F |
|---|---|---|---|---|
| A,G | 3,8 | 3,8 | 8,3 | 8,3 |
| A,H | 3,8 | 3,8 | 8,3 | 8,3 |
| B,G | 5,5 | 2,10 | 5,5 | 2,10 |
| B,H | 5,5 | 1,0 | 5,5 | 1,0 |

- Any pair of pure strategies uniquely identifies a **terminal node**, which identifies a **utility** for each agent (**why?**)
- We have now defined a set of **agents**, **pure strategies**, and **utility functions**
- Any extensive form game defines a corresponding **induced normal form game**

# Reusing Old Definitions

- We can plug our new definition of **pure strategy** into our existing definitions for:
  - Mixed strategy
  - Best response
  - Nash equilibrium (both pure and mixed strategy)

**Question:**

What is the definition of a **mixed strategy** in an extensive form game?

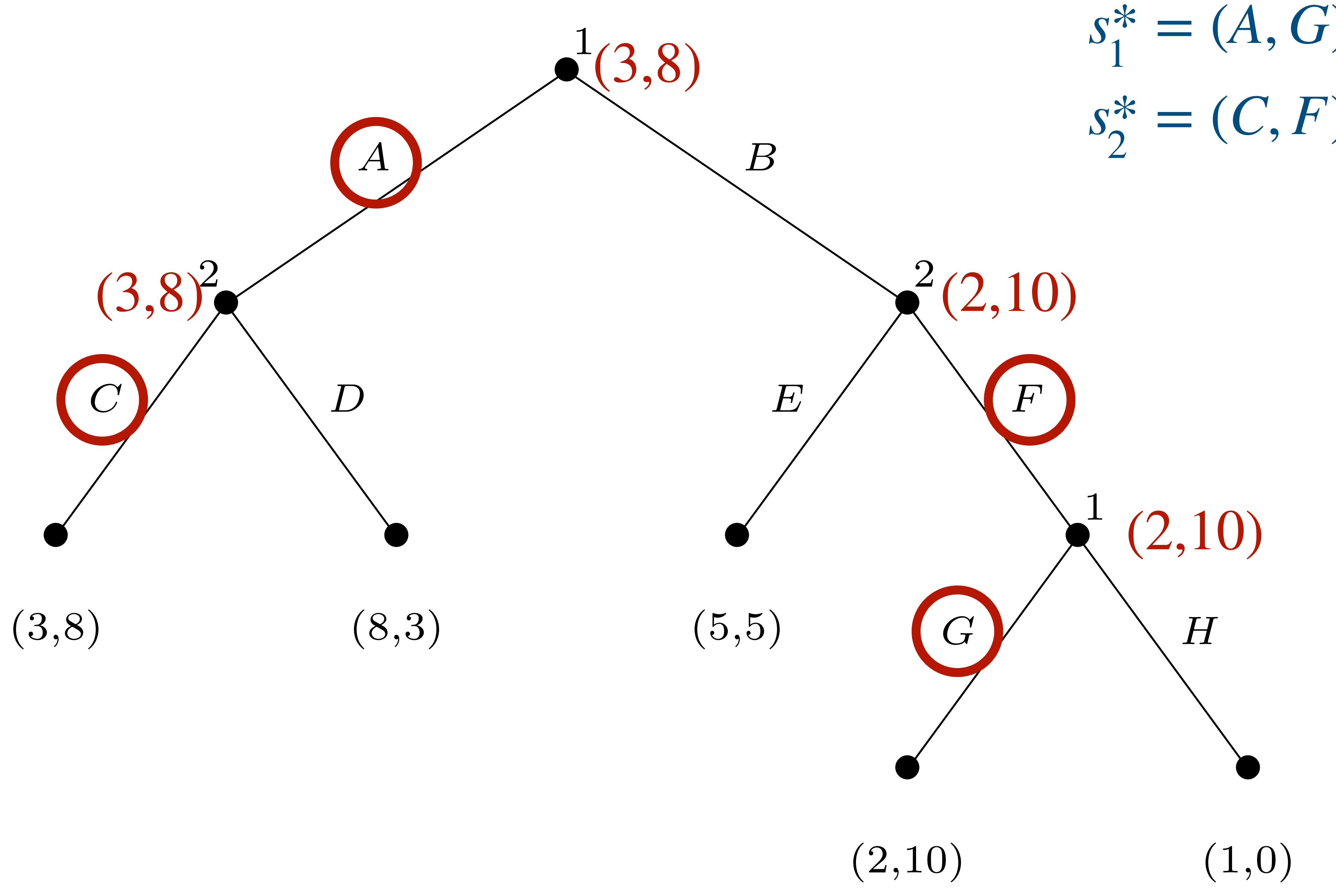
# Pure Strategy Nash Equilibria

**Theorem:** [Zermelo, 1913]
Every finite perfect-information game in extensive form has at least one **pure strategy Nash equilibrium**.

- Starting from the bottom of the tree, no agent needs to **randomize**, because they already know the best response
- There might be **multiple** pure strategy Nash equilibria in cases where an agent has multiple best responses at a **single choice node**

# Backward Induction

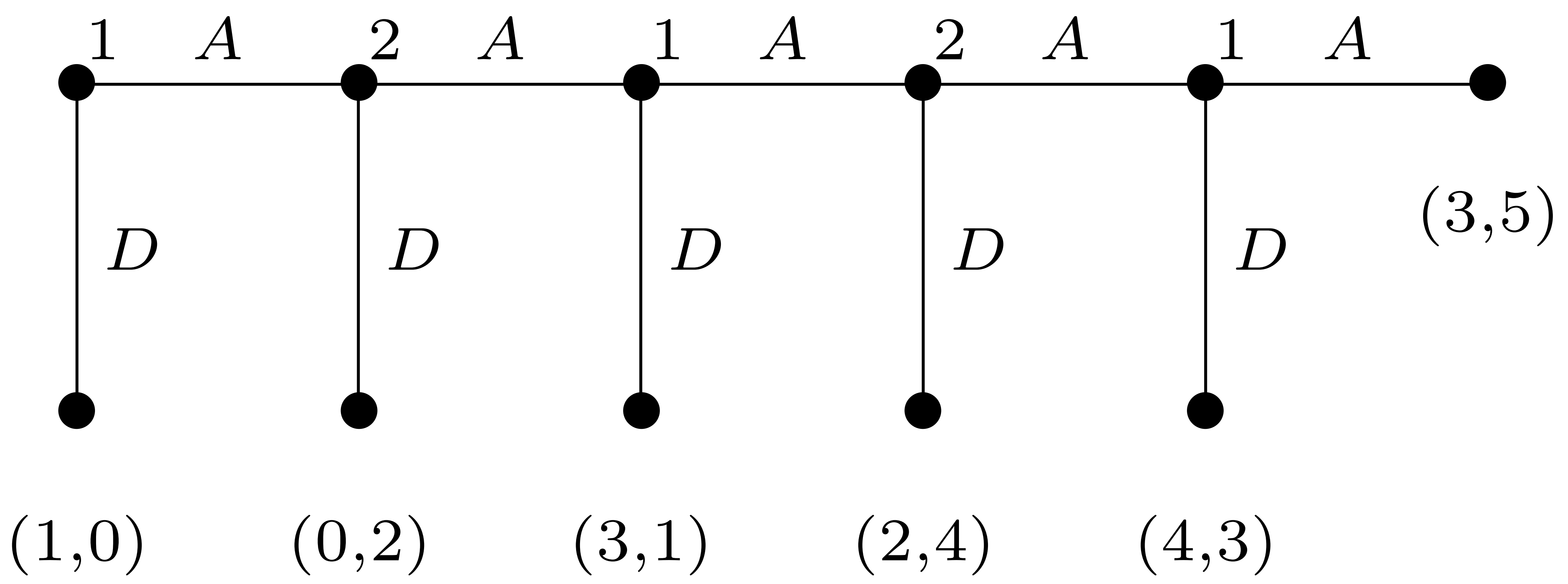
- **Backward induction** is a straightforward algorithm that is guaranteed to compute a pure strategy Nash equilibrium.
- **Idea:** Replace subgames lower in the tree with their equilibrium values




$s_1^* = (A, G)$

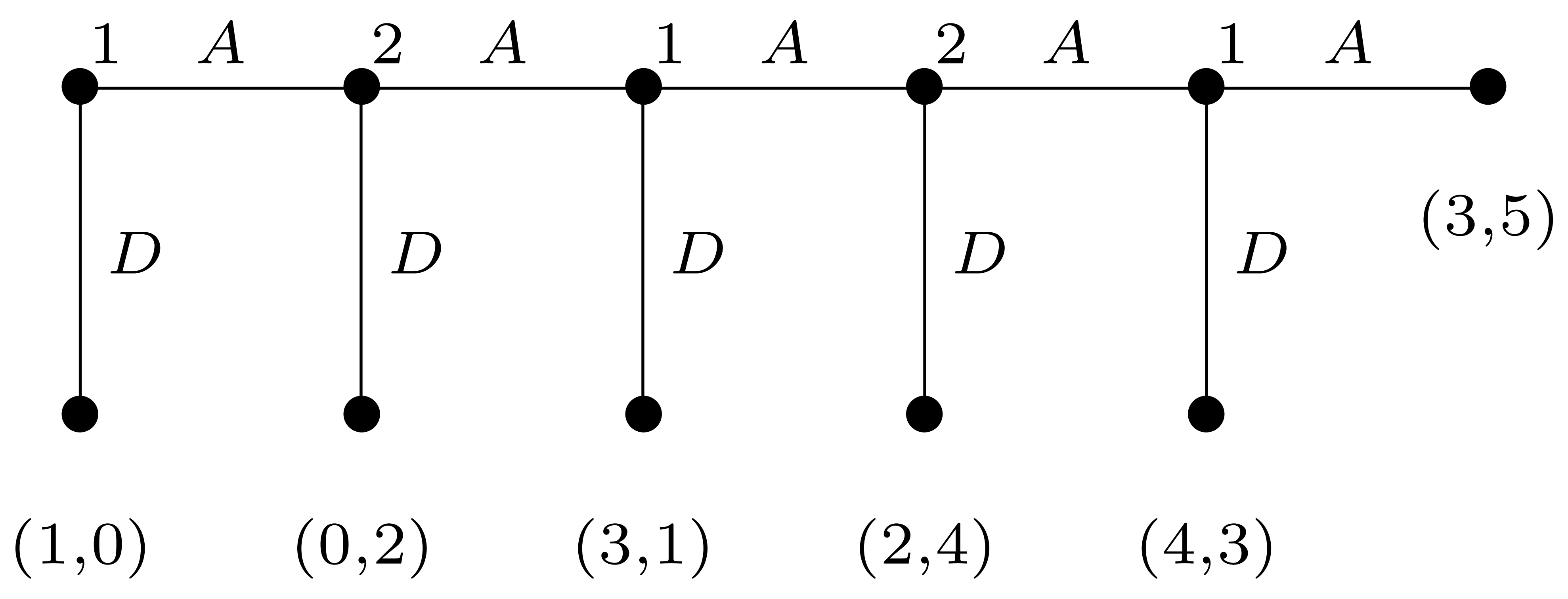
$s_2^* = (C, F)$

# Fun Game: Centipede




- At each stage, one of the players can go **Across** or **Down**
- If they go **Down**, the game ends.

# Backward Induction Criticism




- The **unique** equilibrium is for each player to play **Down at the first opportunity.**
- **Empirically**, this is not how real people tend to play!
- **Theoretically**, what should you do if you arrive at an **off-path** node?
    - How do you update your beliefs to account for this probability 0 event?
    - If player 1 knows that you will update your beliefs in a way that causes you not to play **Down**, then playing **Down** is no longer their only rational choice...

# Summary

- **Mixed strategies** are distributions over **pure strategies**
    - In normal form games, pure strategies are just **single actions**
- **Extensive form games** model **sequential** actions
- **Pure strategies** for extensive form games map **choice nodes** to **actions**
    - **Induced normal form:** normal form game with these pure strategies
    - Notions of mixed strategy, best response, etc. **translate directly**
- **Perfect information:** Every agent **sees all actions** of the other players
    - **Backward induction** computes a **pure strategy Nash equilibrium** for any perfect information extensive form game