Independence in Belief Networks

CMPUT 366: Intelligent Systems

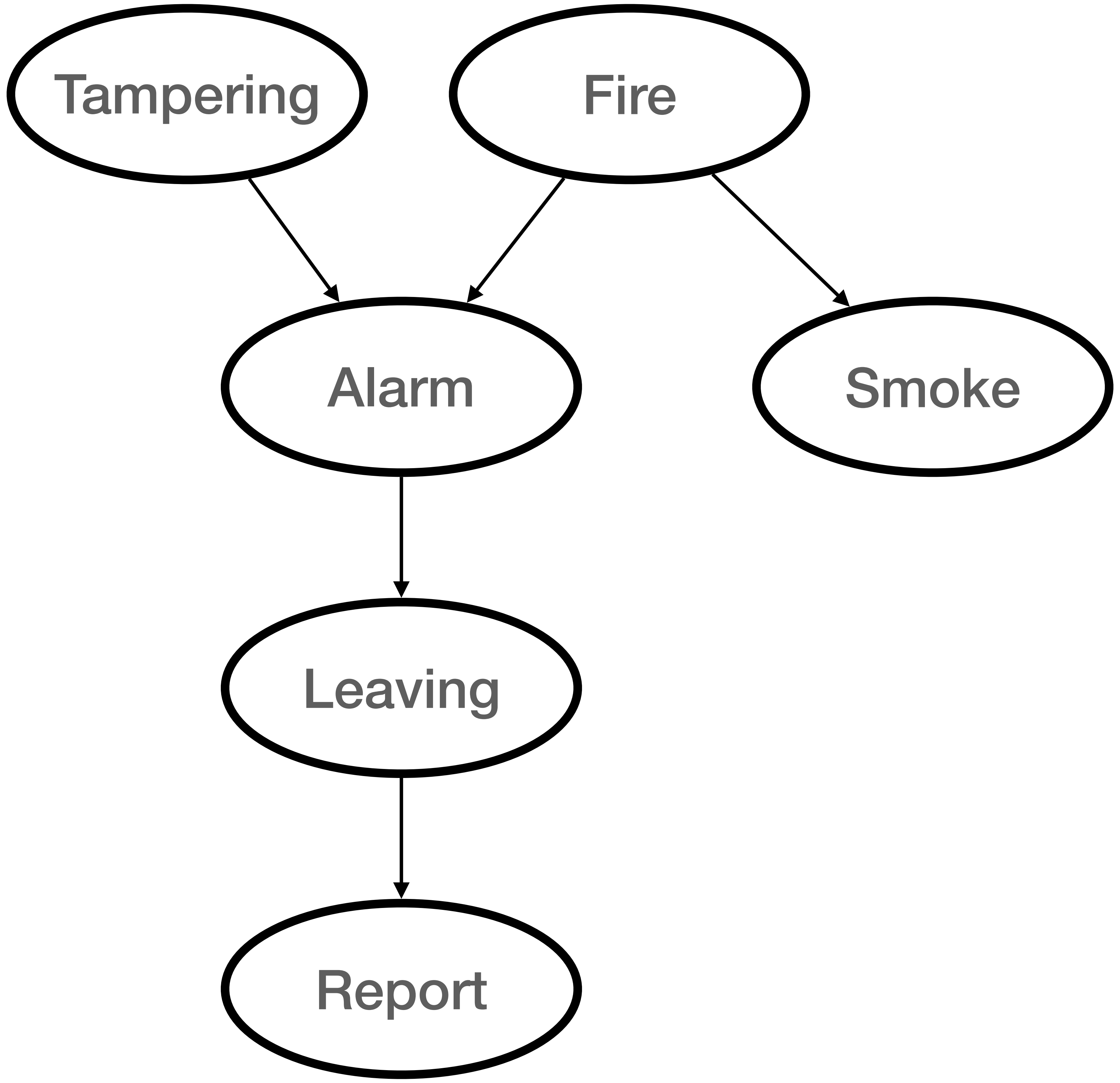
P&M §8.4

Lecture Outline

1. Recap
2. Belief Networks as Factorings
3. Independence in Belief Networks

Recap:
Belief Network Semantics

- **Definition:** Graph representation represents a specific **factorization** of the full **joint distribution**
    - Distribution on each node **conditional on its parents**
    - **Marginal distributions** on nodes with no parents
    - **Product** of these distributions is the joint distribution
    - Not every possible factorization is a **correct** factorization
- **Semantics Theorem:**
  Every node is **independent** of its **non-descendants**, **conditional only** on its **parents**



Recap: Mechanically Constructing Belief Networks

Given a **joint distribution** we can mechanically construct a **correct** encoding:

1. Order the variables $X_1, X_2, \ldots, X_n$ and associate each one with a **node**

2. For each variable X_i:

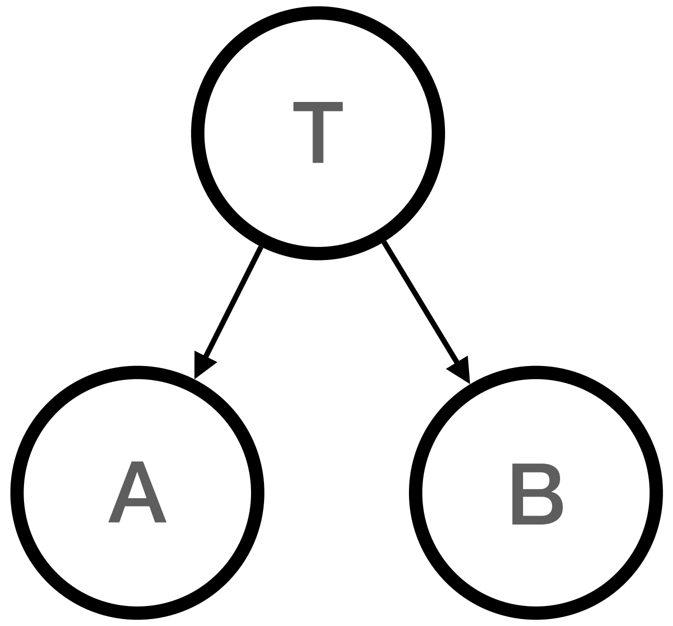
   (i) Choose a **minimal** set of variables $parents(X_i)$ from $X_1, \ldots, X_{i-1}$ such that $P(X_i \mid parents(X_i)) = P(X_i \mid X_1, \ldots, X_{i-1})$

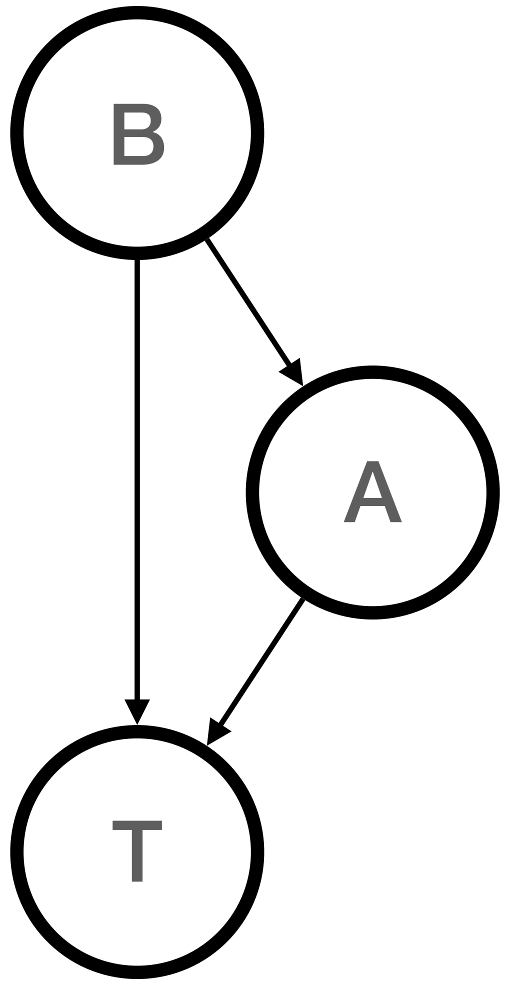
   (ii) i.e., **conditional** on $parents(X_i)$, X_i is **independent** of all the other variables that are **earlier** in the ordering

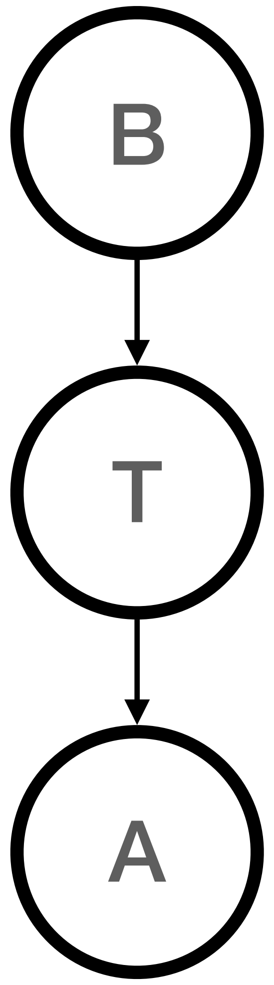
   (iii) Add an **arc** from each variable in $parents(X_i)$ to X_i

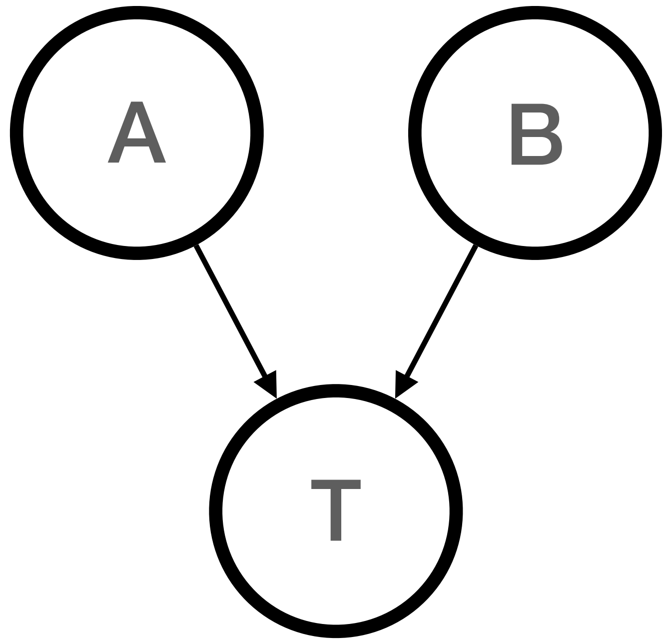
   (iv) Label the node for X_i with the **conditional probability table** $P(X_i \mid parents(X_i))$

Belief Networks as Factorings









- A joint distribution can be factored in **multiple** different ways
  - *Every* variable ordering induces at least one correct factoring (**Why?**)
- A belief network represents a **single** factoring
- Some factorings are correct, some are incorrect

Questions:

1. Does applying the **Chain Rule** to a given variable ordering give a **unique** factoring?
2. Does a given variable ordering correspond to a **unique Belief Network**?

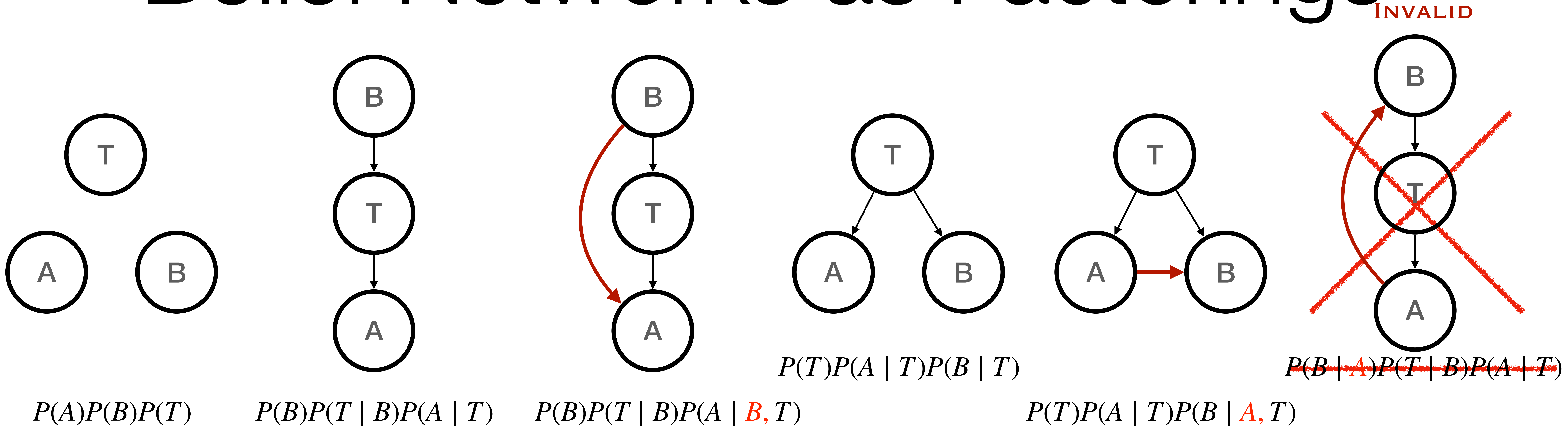
Correct and Incorrect Factorings in the Clock Scenario

Which of the following are **correct** factorings of the joint distribution $P(A, B, T)$ in the Clock Scenario?

1. $P(A)P(B)P(T)$
2. $P(A)P(B \mid A)P(T \mid A, B)$ Chain rule(A,B,T): $P(A)P(B \mid A)P(T \mid A, B)$
3. $P(T)\boxed{P(B \mid T)}P(A \mid T)$ Chain rule(T,B,A): $P(T)\boxed{P(B \mid T, A)}P(A \mid T)$

Which of the above are a **good** factoring for the Clock Scenario? **Why?**

Belief Networks as Factorings

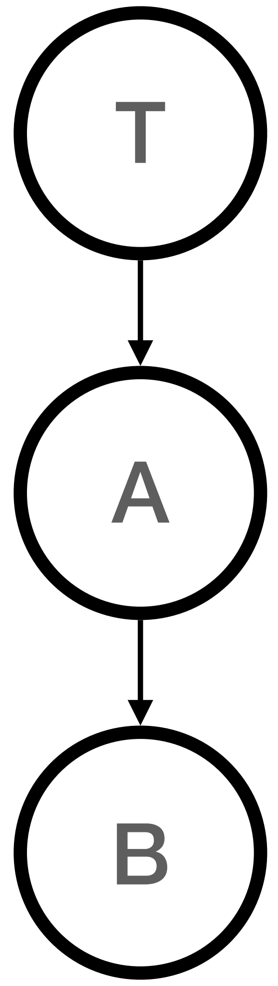


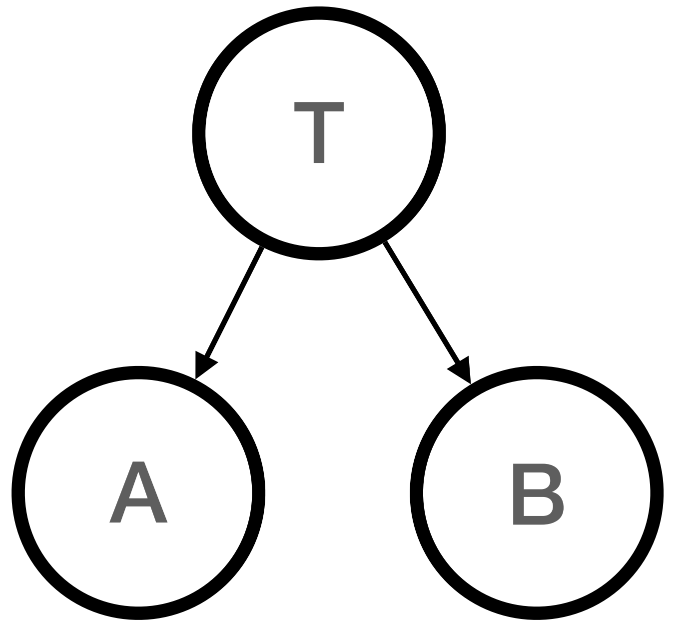
Question: What **factoring** is represented by each network?

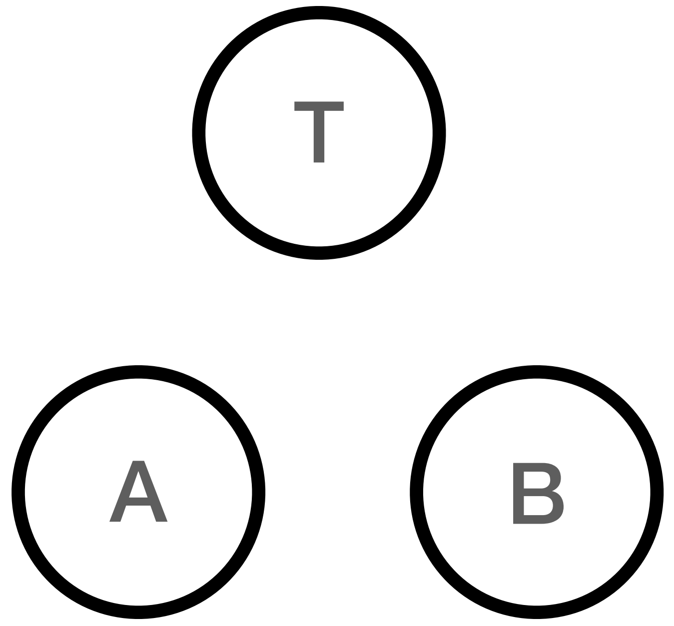
Conditional independence **guarantees** are represented in belief networks by the **absence of edges**.

Variations on the Clock Scenario

- A valid belief is only "correct" or "incorrect" with respect to a given joint distribution
- A **single network** may be correct in one scenario and incorrect in another
- **Telephone Clock Scenario:** Alice looks at the clock, then tells Bob the time over a noisy phone connection
- **Desert Islands Clock Scenario:** Alice is on Island A. Bob is on Island B. The clock is on Island C. Alice and Bob cannot see or hear each other, or the clock.







Independence in a Joint Distribution

Question: How can we answer questions about independence using the **joint distribution**?

Examples using $P(A, B, T)$:

1. Is A independent of B?
   - $P(A = a \mid B = b) = P(A = a)$ **for all** $a \in \text{dom}(A), b \in \text{dom}(B)$?
2. Is T independent of A?
   - $P(T = t \mid A = a) = P(T = t)$ **for all** $a \in \text{dom}(A), t \in \text{dom}(T)$?
3. Is A independent of B given T?
   - $P(A = a \mid B = b, T = t) = P(A = a \mid T = t)$
     for all $a \in \text{dom}(A), b \in \text{dom}(B), t \in \text{dom}(T)$?

$$P(A, B) = \sum_{t \in T} P(A, B, T = t)$$

$$P(A, T) = \sum_{b \in B} P(A, B = b, T)$$

$$P(B, T) = \sum_{a \in A} P(A = a, B, T)$$

$$P(A) = \sum_{b \in B} P(A, B = b)$$

$$P(B) = \sum_{a \in A} P(A = a, B)$$

$$P(T) = \sum_{a \in A} P(A = a, T)$$

$$P(A \mid B, T) = \frac{P(A, B, T)}{P(B, T)}$$

$$P(A \mid B) = \frac{P(A, B)}{P(B)}$$

$$P(A \mid T) = \frac{P(A, T)}{P(T)}$$

$$P(T \mid A) = \frac{P(A, T)}{P(A)}$$

Belief Network Semantics

Definition: A belief network represents a joint distribution that can be factored as

$$P(X_1, \ldots, X_n) = \prod_{i=1}^{n} P(X_i \mid parents(X_i))$$

Theorem: (Belief Network Guarantee)
Every node is **independent** of its **non-descendants**, **conditional only** on its **parents**

Proof:

1. X_j is a descendant of $X_i \implies i < j$
2. For all $i > j$, $P(X_i \mid parents(X_i), X_j) = P(X_i \mid parents(X_i))$
3. For all $i < j$, if j is not a descendant of i, then $P(X_i \mid parents(X_i), X_j) = P(X_i \mid parents(X_i))$

Belief Network Semantics: Proof (2)

$$Z \doteq \{X_1, \ldots, X_{i-1}\} \backslash parents(X_i) \cup \{X_j\}$$
$$= \left\{X_k \mid 1 \le k \le i-1, k \ne i, k \ne j, X_k \notin parents(X_i)\right\}$$

$$P(X_1, \ldots, X_i) = P(X_i \mid parents(X_i))P(X_1, \ldots, X_{i-1}) = P(X_i \mid parents(X_i)) \prod_{k=1}^{i-1} P(X_k \mid parents(X_k))$$ def. belief network

$$P(X_i, X_j, parents(X_i)) = \sum_Z P(X_1, \ldots, X_i) = \sum_Z P(X_i \mid parents(X_i)) \prod_{k=1}^{i-1} P(X_k \mid parents(X_k))$$ marginalization

$$P(X_i \mid parents(X_i), X_j) = \frac{P(X_i, parents(X_i), X_j)}{P(parents(X_i), X_j)}$$ def. conditional probability

$$= \frac{\sum_Z P(X_i \mid parents(X_i)) \prod_{k=1}^{i-1} P(X_k \mid parents(X_k))}{\sum_Z \prod_{k=1}^{i-1} P(X_k \mid parents(X_k))}$$

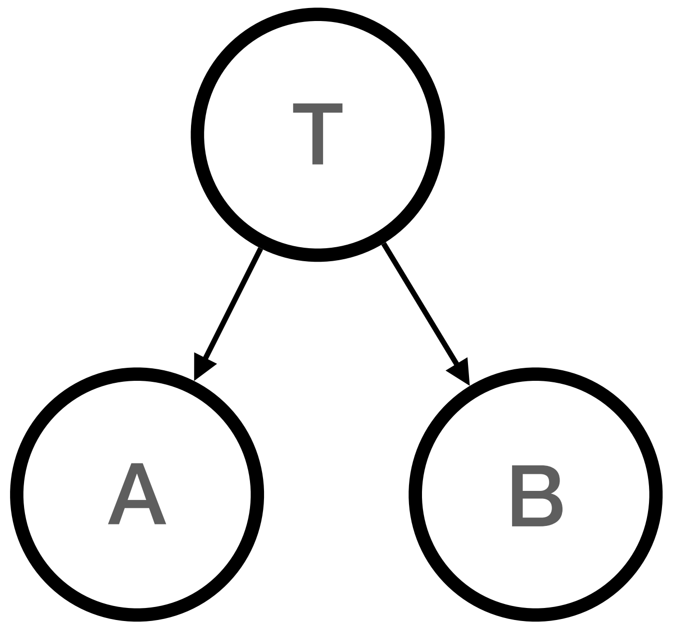
$$= P(X_i \mid parents(X_i)) \underbrace{\boxed{\frac{\sum_Z \prod_{k=1}^{i-1} P(X_k \mid parents(X_k))}{\sum_Z \prod_{k=1}^{i-1} P(X_k \mid parents(X_k))}}}_{=1} = P(X_i \mid parents(X_i)) \blacksquare$$

Independence in a Belief Network

Belief Network Semantics:
Every node is **independent** of its **non-descendants**, **conditional only** on its **parents**

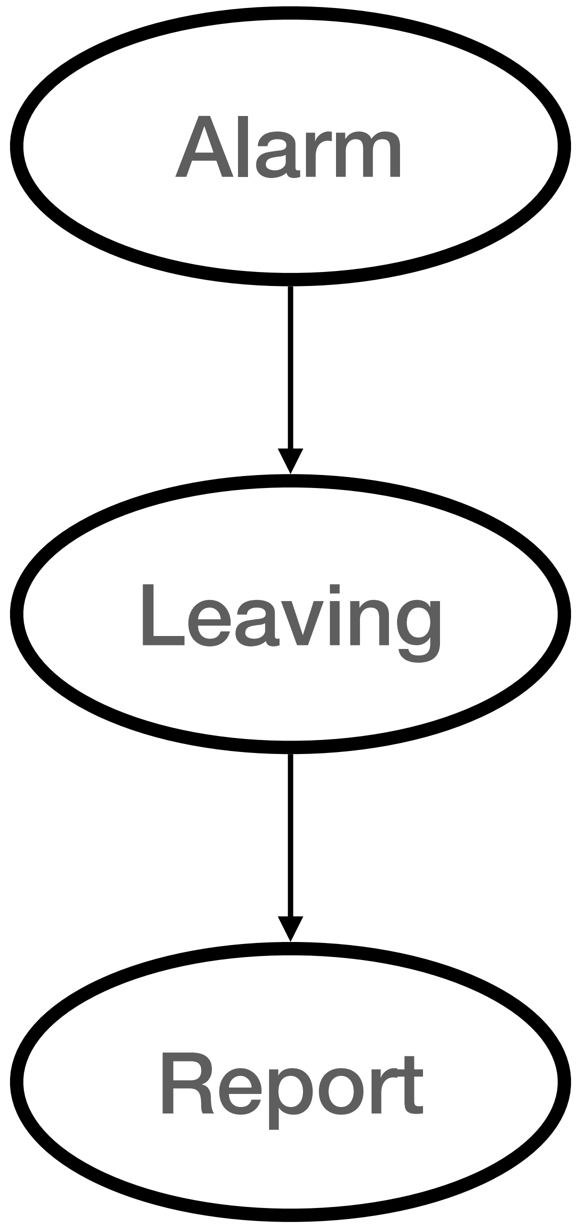
- We can use the semantics of a correct belief network to answer questions about independence
- Examples using the belief network at right:

1. Is **T** independent of **A**?
2. Is **A** independent of **B** given **T**?
3. Is **A** independent of **B**?



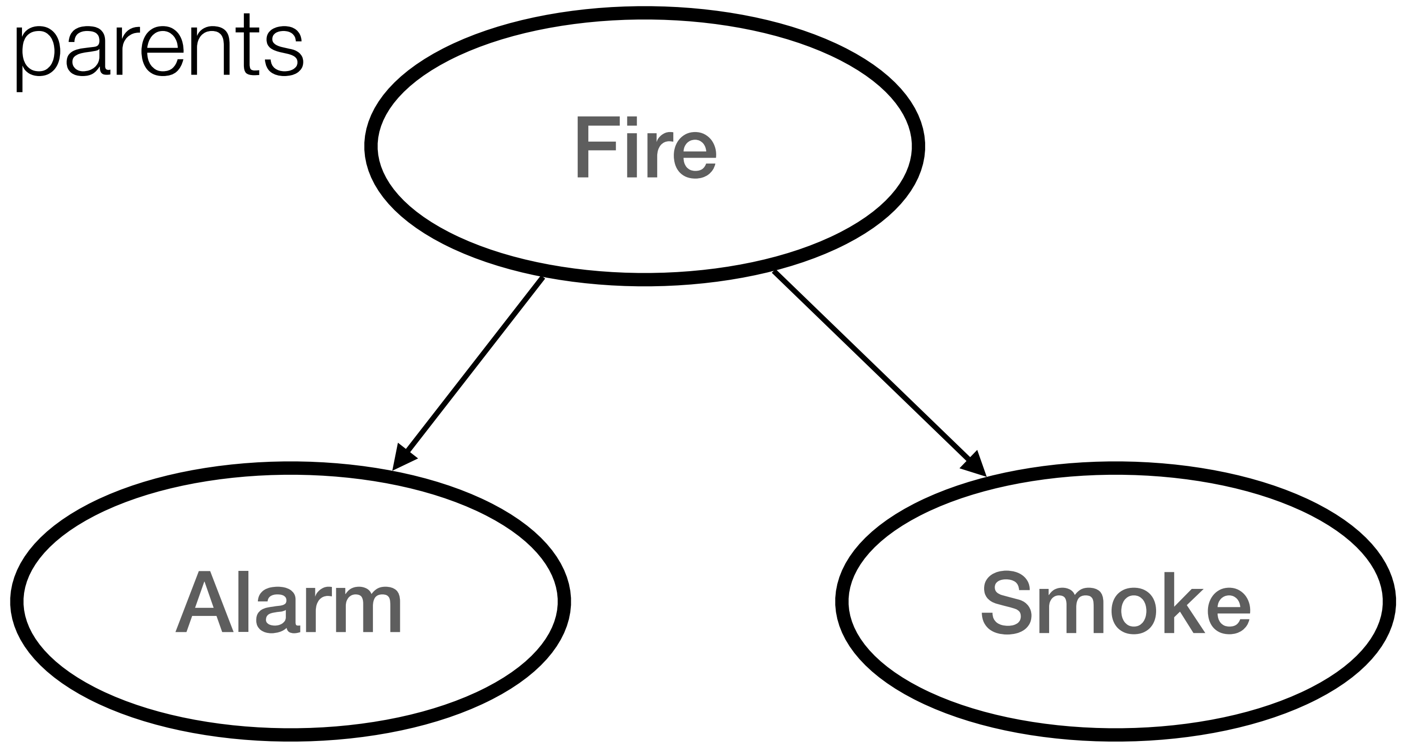
Chain

- **Question:** Is Report independent of Alarm given Leaving?
    - *Intuitively:* The only way learning Report tells us about Alarm is because it tells us about Leaving; but Leaving has already been observed
    - *Formally:* Report is independent of its non-descendants given only its parents
        - Leaving is Report's parent
        - Alarm is a non-descendant of Report
- **Question:** Is Report independent of Alarm?
    - *Intuitively:* Learning Report gives us information about Leaving, which gives us information about Alarm
    - *Formally:* Report is independent of Alarm given Report's parents; but the question is about **marginal** independence



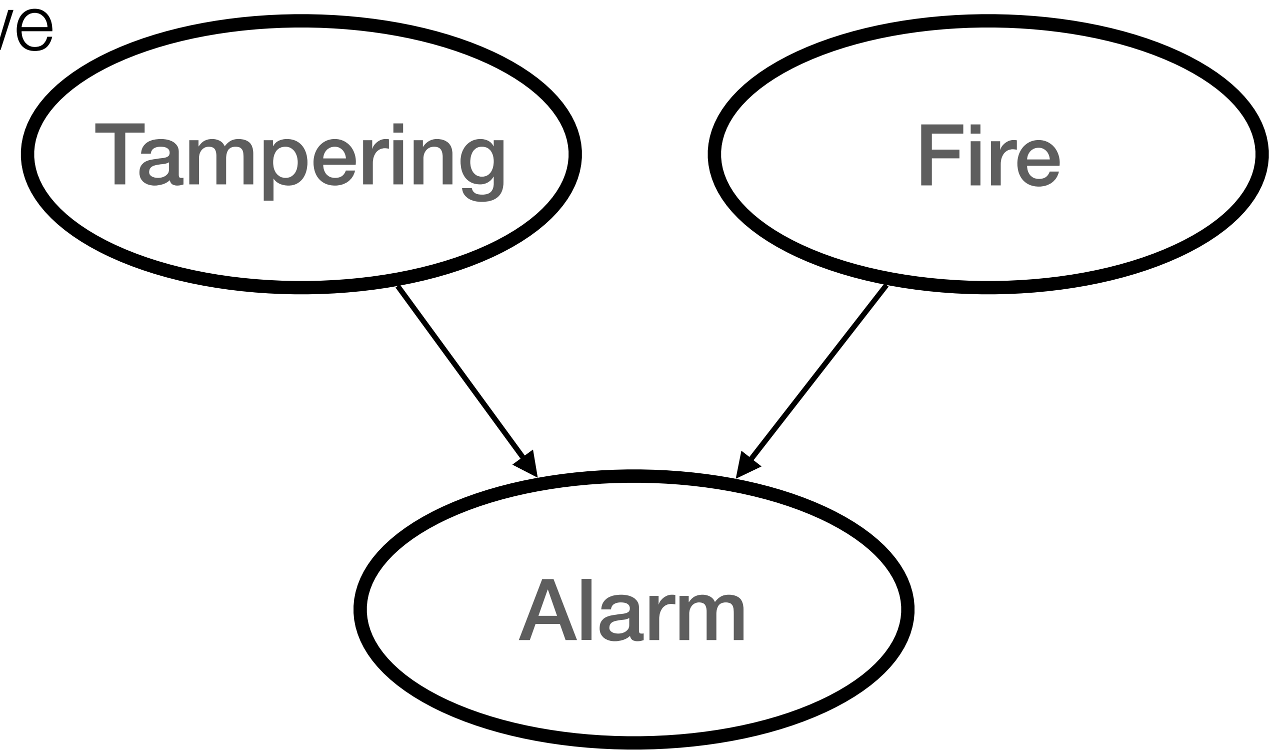
Common Ancestor

- **Question:** Is **Alarm** independent of **Smoke** given **Fire**?
    - *Intuitively:* The only way learning **Smoke** tells us about **Alarm** is because it tells us about **Fire**; but **Fire** has already been observed
    - *Formally:* **Alarm** is independent of its non-descendants given only its parents
        - **Fire** is **Alarm's** parent
        - **Smoke** is a non-descendant of **Alarm**
- **Question:** Is **Alarm** independent of **Smoke**?
    - *Intuitively:* Learning **Smoke** gives us information about **Fire**, which gives us information about **Alarm**
    - *Formally:* **Alarm** is independent of **Smoke** given only **Alarm's** parents; but the question is about **marginal independence**



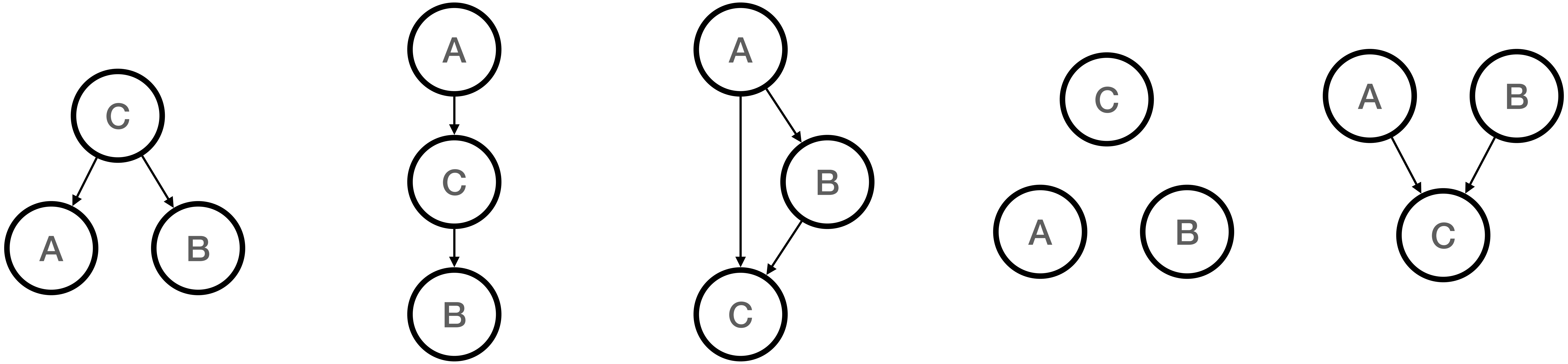
Common Descendant

- **Question:** Is Tampering independent of Fire given Alarm?
  - *Intuitively:* If we know Alarm is ringing, then both Tampering and Fire are more likely. If we then learn that Fire is false, that makes it more likely that the Alarm is ringing because of Tampering.
  - *Formally:* Tampering is independent of Fire given **only** Tampering's parents; but we are conditioning on one of Tampering's **descendants**
    - Conditioning on a **common descendant** can make independent variables dependent through this **explaining away** effect
- **Question:** Is Tampering (marginally) independent of Fire?
  - *Intuitively:* Learning Tampering doesn't tell us anything about whether a Fire is happening
  - *Formally:* Tampering is independent of Fire given Tampering's parents
    - Tampering has no parents, so we are always conditioning on them
    - Fire is a non-descendant of Tampering



Correctness of a Belief Network

A belief network is a **correct** representation of a joint distribution when the belief network answers "yes" to an independence question **only if** the **joint distribution** answers "yes" to the same question.



Questions:

1. Is A independent of B in the above belief networks?
2. Is A independent of B given C in the above belief networks?

Summary

- A belief network represents a specific **factoring** of a joint distribution
    - More than one belief network can correctly represent a joint distribution
    - A given belief network may be correct for one underlying joint distribution and incorrect for another
- A **good** belief network is one that encodes as many **true** conditional independence relationships as possible
- It is possible to read the conditional independence guarantees made by a belief network directly from its **graph structure**