# Function Approximation & Policy Gradient Methods

CMPUT 261: Introduction to Artificial Intelligence

S&B §9.0-9.5.4, 13.0-13.3

# Lecture Outline

1. Recap & Logistics
2. Parameterized Value Functions
3. Gradient Descent
4. Approximation Schemes
5. Parameterized Policies
6. Policy Gradient Theorem
7. REINFORCE algorithm

*After this lecture, you should be able to:*

- explain why function approximation is useful
- define tile coding
- explain the difference between action-value and policy gradient methods for control
- state the Policy Gradient Theorem and explain why it is important
- trace an execution of the REINFORCE algorithm

# Logistics

- **Assignment #4** is due **April 11** at 11:59pm
    - Late submissions for 20% deduction until **April 13** at 11:59pm
- **SPOT** (formerly USRI) surveys are now available: https://p20.courseval.net/etw/ets/et.asp?nxappid=UA2&nxmid=start
    - Available until **April 14**
    - You should have gotten an email
    - Please do fill one out for this class!

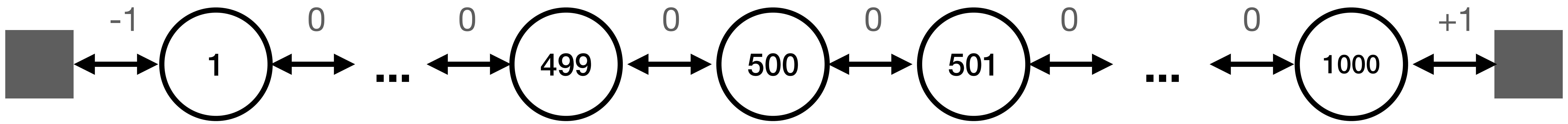
# Recap: TD Learning

- Temporal Difference Learning **bootstraps** *and* learns from **experience**
    - Dynamic programming bootstraps, but doesn't learn from experience (requires full dynamics)
    - Monte Carlo learns from experience, but doesn't bootstrap
- Prediction: **TD(0) algorithm**
- **Sarsa** estimates action-values of **actual $\epsilon$-greedy policy**
- **Q-Learning** estimates action-values of **optimal** policy while **executing** an **$\epsilon$-greedy** policy

# Tabular Value Functions

- We have been assuming a **tabular representation** for value function estimates $V(s)$ and $Q(s, a)$
    - We can **separately** set the value of $V(s)$ or $Q(s, a)$ for every possible $s \in \mathcal{S}$ and $a \in \mathcal{A}$
- This implicitly means that we **must** store a separate value for every possible input for the value function
- **Question:** What should we do if there are **too many states** to store a value for each? (e.g., **pixel values** in the Atari setting)
- **Question:** What should we do if the state **isn't fully observable**?

# Example: Number Line Walk



$$\pi(a \mid s) = 0.5 \quad \forall s \in \mathcal{S}, a \in \{\text{left}, \text{right}\}$$

- **Question:** Would dynamic programming, Monte Carlo, or TD(0) work to estimate $v_\pi$?
- **Question:** How much **storage** would that require?
- **Question:** What could we do instead?

(Image: Sutton & Barto, 2018)

# Parameterized Value Functions

- A **parameterized value function**'s values are set by setting the values of a **weight vector** $\mathbf{w} \in \mathbb{R}^d$:

$$\hat{v}(s, \mathbf{w}) \approx v_\pi(s)$$

  - $\hat{v}$ could be a **linear function**: $\mathbf{w}$ is feature weights for state **features** $\mathbf{x}(s)$
  - $\hat{v}$ could be a **neural network**: $\mathbf{w}$ is weights, biases, kernels, etc.
- Many fewer weights than states: $d \ll |\mathcal{S}|$
  - Changing **one weight** changes the estimated value of **many states**
  - Updating a single state **generalizes** to affect many other states' values

# Decoupled Estimates

- With **tabular** estimates:
  - Can update the value of a single state **individually**
  - Estimates can be **exactly correct** for **each state**
- For **parameterized** estimates:
  - Estimates cannot necessarily be correct for each state (e.g., when two states have identical features but different values)
  - Cannot independently adjust state values

# Prediction Objective

- Since we cannot guarantee that every state will be correct, we must **trade off** estimation quality of one state vs. another
- We will use a distribution $\mu(s)$ to specify **how much we care** about the quality of our value estimate for each state
- We will optimize the **mean squared value error**:

$$\overline{VE}(\mathbf{w}) \doteq \sum_{s \in \mathcal{S}} \mu(s) \left[ v_\pi(s) - \hat{v}(s, \mathbf{w}) \right]^2$$

- **Note:** If we knew $v_\pi$, this would be a **supervised learning problem** with a **loss** of $\overline{VE}$
- **Question:** What should we use for $\mu(s)$?

# Stochastic Gradient Descent with Known True Values

- Suppose we are given a **new example**: $\left(S_t, v_\pi(S_t)\right)$
- How should we update our weight vector $\mathbf{w}$?
- **Stochastic Gradient Descent:** After each example, adjust weights a tiny bit in **direction** that would most **reduce error** on **that example**:

$$\mathbf{w}_{t+1} \doteq \mathbf{w}_t - \frac{1}{2}\alpha \nabla \left[\overset{\text{target}}{\boxed{v_\pi(S_t)}} - \hat{v}(S_t, \mathbf{w}_t)\right]^2$$

$$= \mathbf{w}_t - \frac{1}{2}\alpha \nabla \left[(v_\pi(S_t))^2 - 2v_\pi(S_t)\hat{v}(S_t, \mathbf{w}_t) + (\hat{v}(S_t, \mathbf{w}_t))^2\right]$$

$$= \mathbf{w}_t + \alpha \left[v_\pi(S_t) - \hat{v}(S_t, \mathbf{w}_t)\right] \nabla \hat{v}(s, \mathbf{w}_t)$$

# Stochastic Gradient Descent with Unknown True Values

- If we knew $v_\pi(s)$, we would be done!
- Instead, we will update toward an **approximate target** $U_t$:

$$\mathbf{w}_{t+1} \leftarrow \mathbf{w}_t + \alpha \left[ U_t - \hat{v}(S_t, \mathbf{w}_t) \right] \nabla \hat{v}(s, \mathbf{w}_t)$$

- $U_t$ can be any of our update targets from previous lectures

# Gradient Monte Carlo

- **Monte Carlo target:** $U_t = G_t$
- $U_t$ is an unbiased estimate of $v_\pi(S_t)$: $\mathbb{E}[U_t \mid S_t = s] = v_\pi(s)$

**Gradient Monte Carlo Algorithm for Estimating $\hat{v} \approx v_\pi$**

```
Input: the policy $\pi$ to be evaluated
Input: a differentiable function $\hat{v} : \mathcal{S} \times \mathbb{R}^d \to \mathbb{R}$
Algorithm parameter: step size $\alpha > 0$
Initialize value-function weights $\mathbf{w} \in \mathbb{R}^d$ arbitrarily (e.g., $\mathbf{w} = \mathbf{0}$)

Loop forever (for each episode):
    Generate an episode $S_0, A_0, R_1, S_1, A_1, \ldots, R_T, S_T$ using $\pi$
    Loop for each step of episode, $t = 0, 1, \ldots, T-1$:
        $\mathbf{w} \leftarrow \mathbf{w} + \alpha\big[G_t - \hat{v}(S_t,\mathbf{w})\big]\nabla\hat{v}(S_t,\mathbf{w})$
```

## Semi-gradient

- **TD(0) target:** $U_t = R_{t+1} + \gamma \hat{v}(S_{t+1}, \mathbf{w}_t)$
- Bootstrapping targets like TD(0) depend on the **current value** of $\mathbf{w}_t$, so they are **not unbiased**
- Gradient $\nabla \hat{v}(s, \mathbf{w}_t)$ accounts for change in the **estimate** from change in $\mathbf{w}_t$
- But updates to $\mathbf{w}$ change both the **estimate** *and* the **target**
- We call these updates **semi-gradient** updates

# Semi-gradient TD(0)

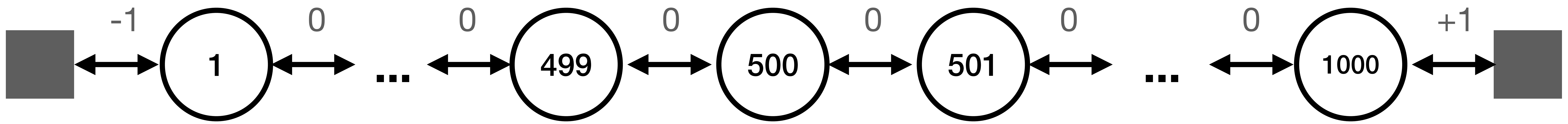
- **TD(0) target:** $U_t = R_{t+1} + \gamma \hat{v}(S_{t+1}, \mathbf{w}_t)$

**Semi-gradient TD(0) for estimating $\hat{v} \approx v_\pi$**

```
Input: the policy $\pi$ to be evaluated
Input: a differentiable function $\hat{v} : \mathcal{S}^+ \times \mathbb{R}^d \rightarrow \mathbb{R}$ such that $\hat{v}(\text{terminal},\cdot) = 0$
Algorithm parameter: step size $\alpha > 0$
Initialize value-function weights $\mathbf{w} \in \mathbb{R}^d$ arbitrarily (e.g., $\mathbf{w} = \mathbf{0}$)

Loop for each episode:
    Initialize $S$
    Loop for each step of episode:
        Choose $A \sim \pi(\cdot|S)$
        Take action $A$, observe $R, S'$
        $\mathbf{w} \leftarrow \mathbf{w} + \alpha\big[R + \gamma\hat{v}(S',\mathbf{w}) - \hat{v}(S,\mathbf{w})\big]\nabla\hat{v}(S,\mathbf{w})$
        $S \leftarrow S'$
    until $S$ is terminal
```

# State Aggregation



$$\pi(a \mid s) = 0.5 \quad \forall s \in \mathcal{S}, a \in \{\text{left}, \text{right}\}$$

- One easy way to reduce the memory usage for a large state space is to **aggregate** states together
- In the Number Line Walk example, we could group the states into 10 groups of 100 states each
- $\mathbf{w}$ is a 10-element vector
- $\hat{v}(s, \mathbf{w}) = \mathbf{w}_{x(s)}$, where $x(s) = \left\lfloor \frac{s}{100} \right\rfloor$

(Image: Sutton & Barto, 2018)

# State Aggregation Performance

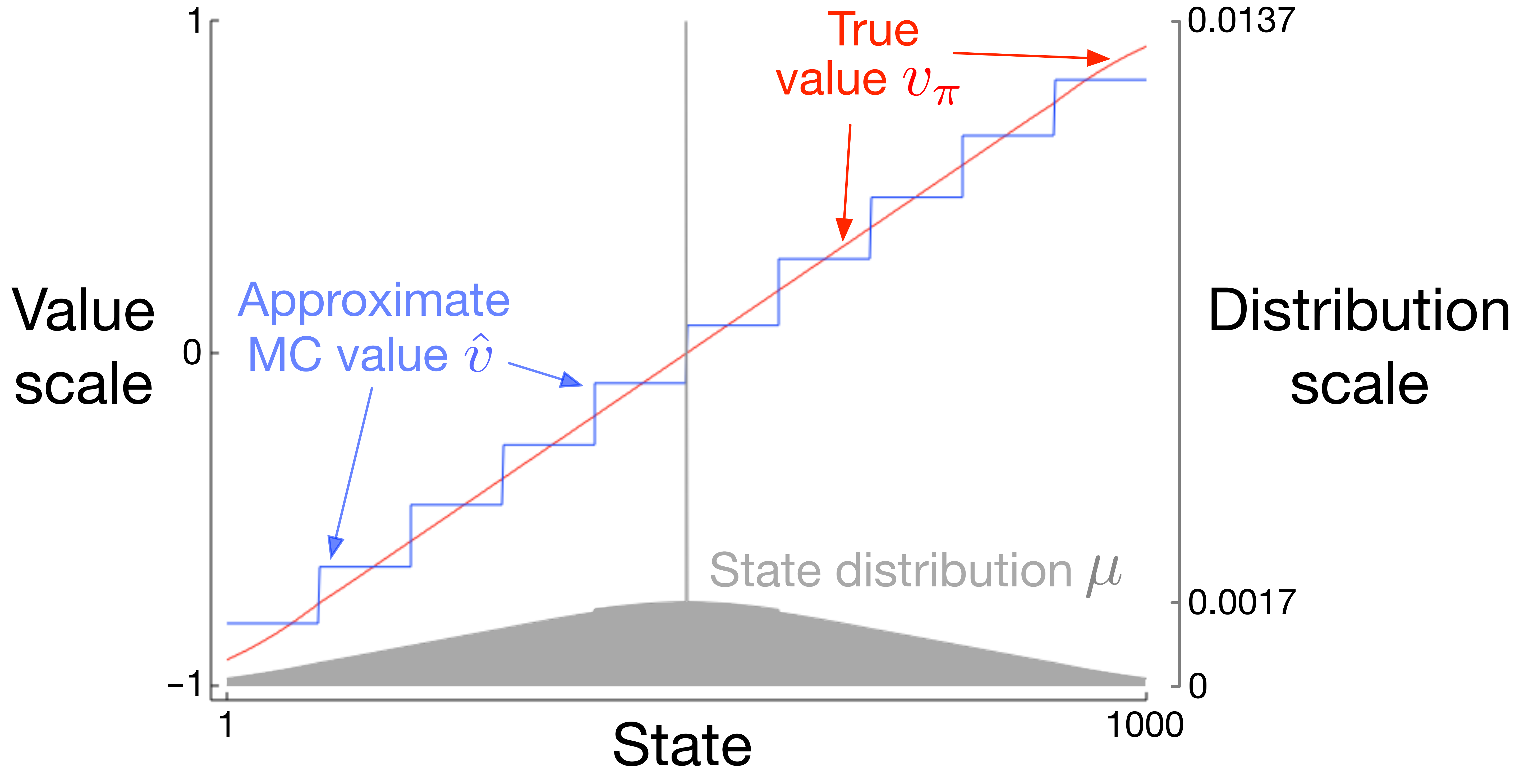


**Figure 9.1:** Function approximation by state aggregation on the 1000-state random walk task, using the gradient Monte Carlo algorithm (page 202).

(Image: Sutton & Barto, 2018)

# Linear Approximation

- Every state $s \in \mathcal{S}$ is assigned a **feature vector** $\mathbf{x}(s)$

$$\mathbf{x}(s) \doteq (x_1(s), x_2(s), \ldots, x_d(s))$$

- State-value function approximation:

$$\hat{v}(s, \mathbf{w}) \doteq \mathbf{w}^T \mathbf{x}(s) = \sum_{i=1}^{d} w_i x_i(s)$$

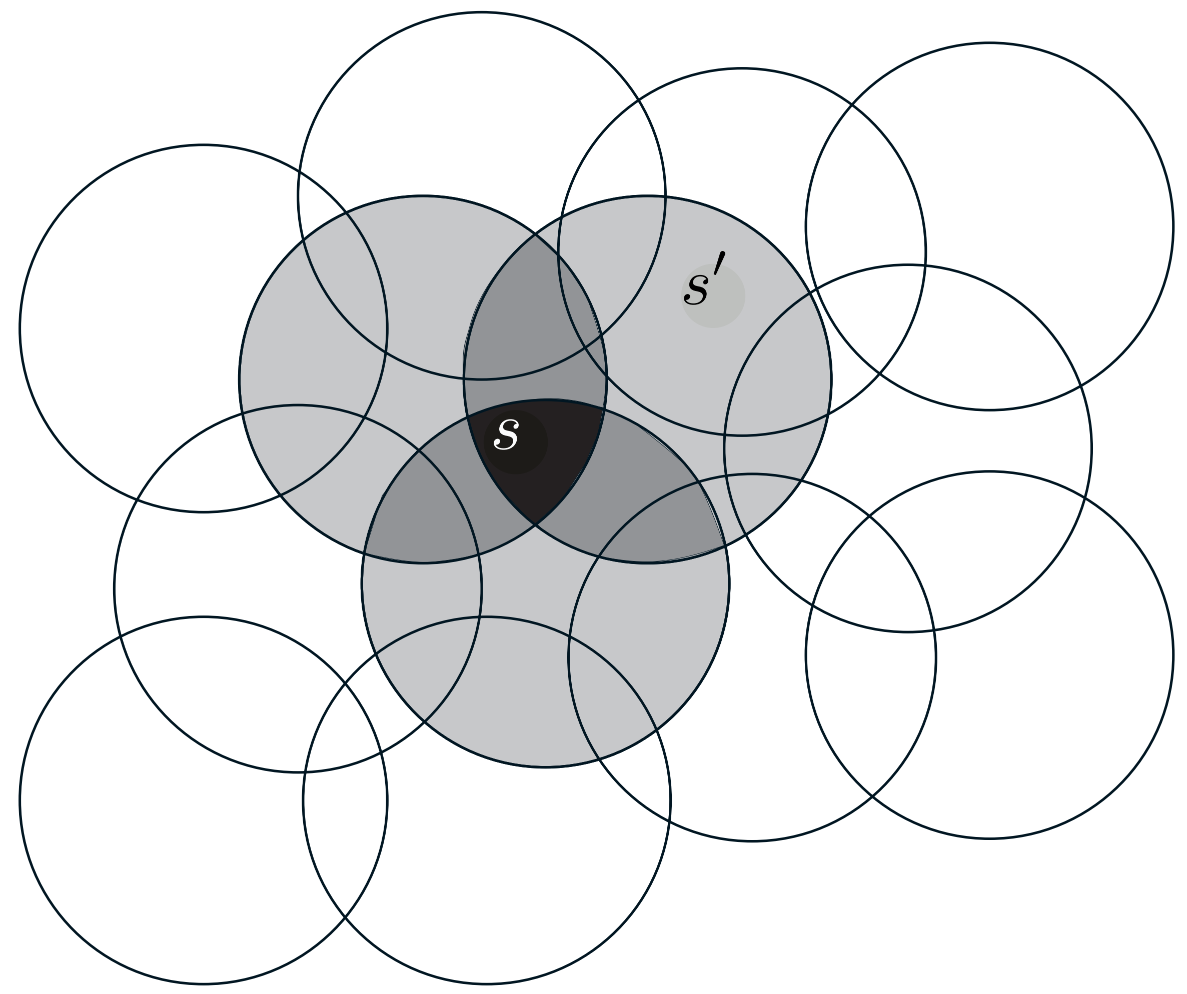
- **Gradient** is easy: $\nabla \hat{v}(s, \mathbf{w}) = \mathbf{x}(s)$

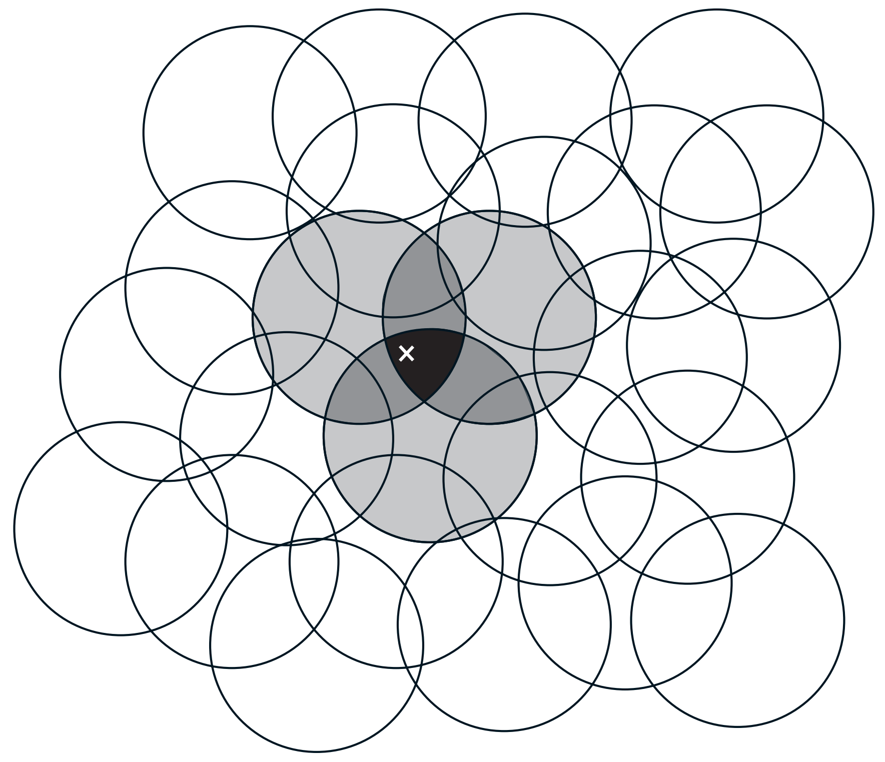
- **Gradient updates** are easy: $\mathbf{w}_{t+1} \leftarrow \mathbf{w}_t + \alpha \left[ U_t - \hat{v}(s, \mathbf{w}_t) \right] \mathbf{x}(s)$

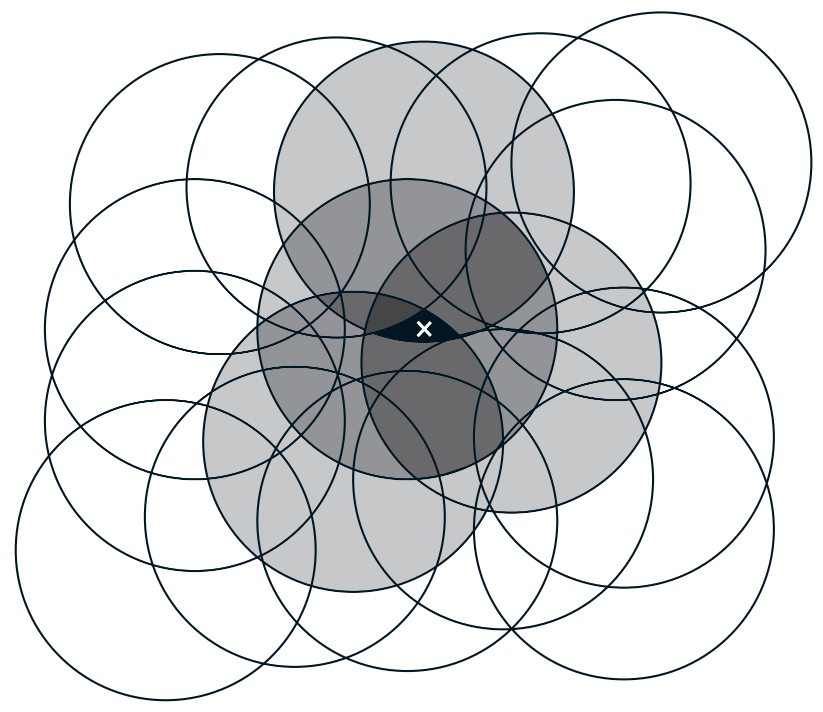
- State aggregation is a **special case** of linear approximation (**why?**)

# Feature Construction: Coarse Coding

- Divide state space up into **overlapping cells**
- One **indicator feature** for each cell, set to 1 if the state is in the cell
- This is another form of **state aggregation**
- Updating one state **generalizes** to other states that **share a cell**



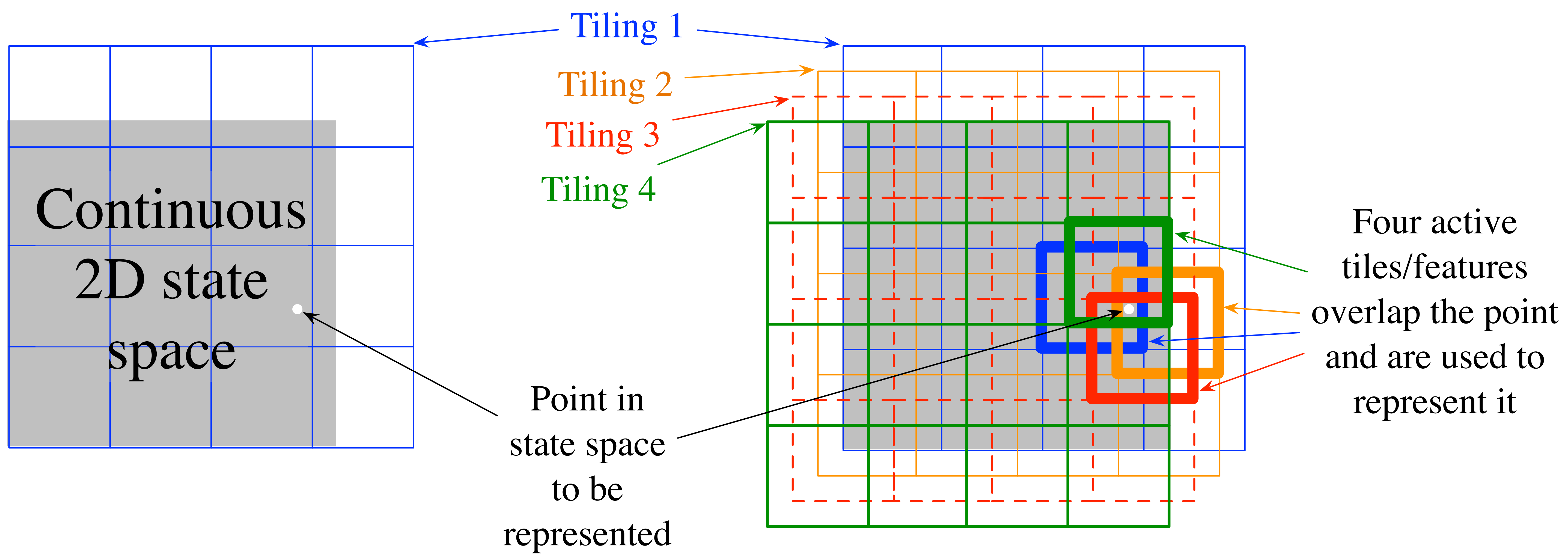

Narrow generalization


Broad generalization

(Image: Sutton & Barto, 2018)

# Tile Coding

- The most practical form of coarse coding
- Partition state space into a uniform grid called a **tiling**
    - Use **multiple** tilings that are **offset**


Tiling 1
Tiling 2
Tiling 3
Tiling 4
Continuous 2D state space
Point in state space to be represented
Four active tiles/features overlap the point and are used to represent it

(Image: Sutton & Barto, 2018)

# Approaches to Control

1. **Action-value methods** (all previous approaches)
    - Learn the value of **each action** in **each state**: $q_\pi(s, a)$
    - Pick the **max-value action** (usually): $\arg\max_a q_\pi(s, a)$
2. **Function approximation** (just now)
    - **Prediction:** Learn the **parameters** $\mathbf{w}$ of state-value function $\hat{v}(s, \mathbf{w})$
    - **Control:** Learn the **parameters** $\mathbf{w}$ of action-value function $\hat{q}(s, \mathbf{w})$
3. **Policy-gradient methods** (rest of today)
    - Learn the **parameters** $\theta$ of a **policy** $\pi(a \mid s, \theta)$
    - Update by **gradient ascent** in performance

# Parameterized Policies

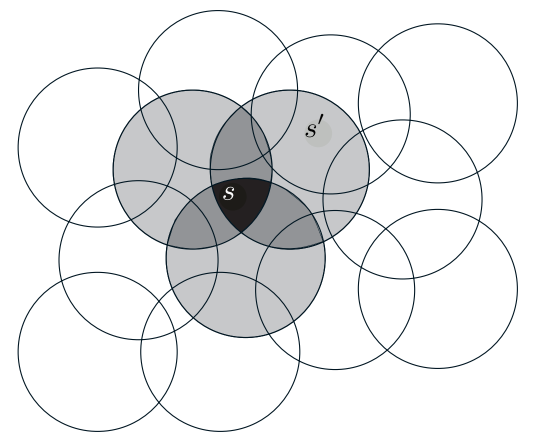
- The action probabilities of a **parameterized policy** $\pi(a \mid s, \theta)$ are set by setting the values of a **parameter vector** $\theta \in \mathbb{R}^{d'}$
- Common approach: **softmax in action preferences**
  - Learn an **action preference function** $h(s, a, \theta)$
  - **Softmax** over action preferences gives action probabilities:

$$\pi(a \mid s, \theta) \doteq \frac{e^{h(s,a,\theta)}}{\sum_{a'} e^{h(s,a',\theta)}}$$

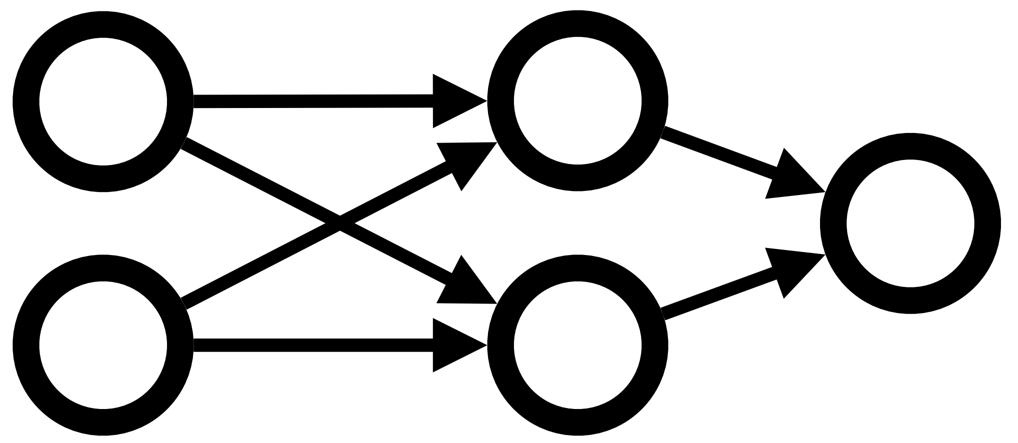
# Action Preferences

- **Question:** What **functional forms** can we use for action preferences?
- Anything we could have used for $\hat{v}$:
  - **Linear approximations:**

$$h(s, a, \theta) \doteq \theta^T \mathbf{x}(s) = \sum_{i=1}^{d} \theta_i x_i(s)$$



    - Including state aggregation, coarse coding, tile coding
  - **Neural network:** $\theta$ are weights, offsets, kernels, etc.



# Parameterized Policies Advantage: Deterministic Action

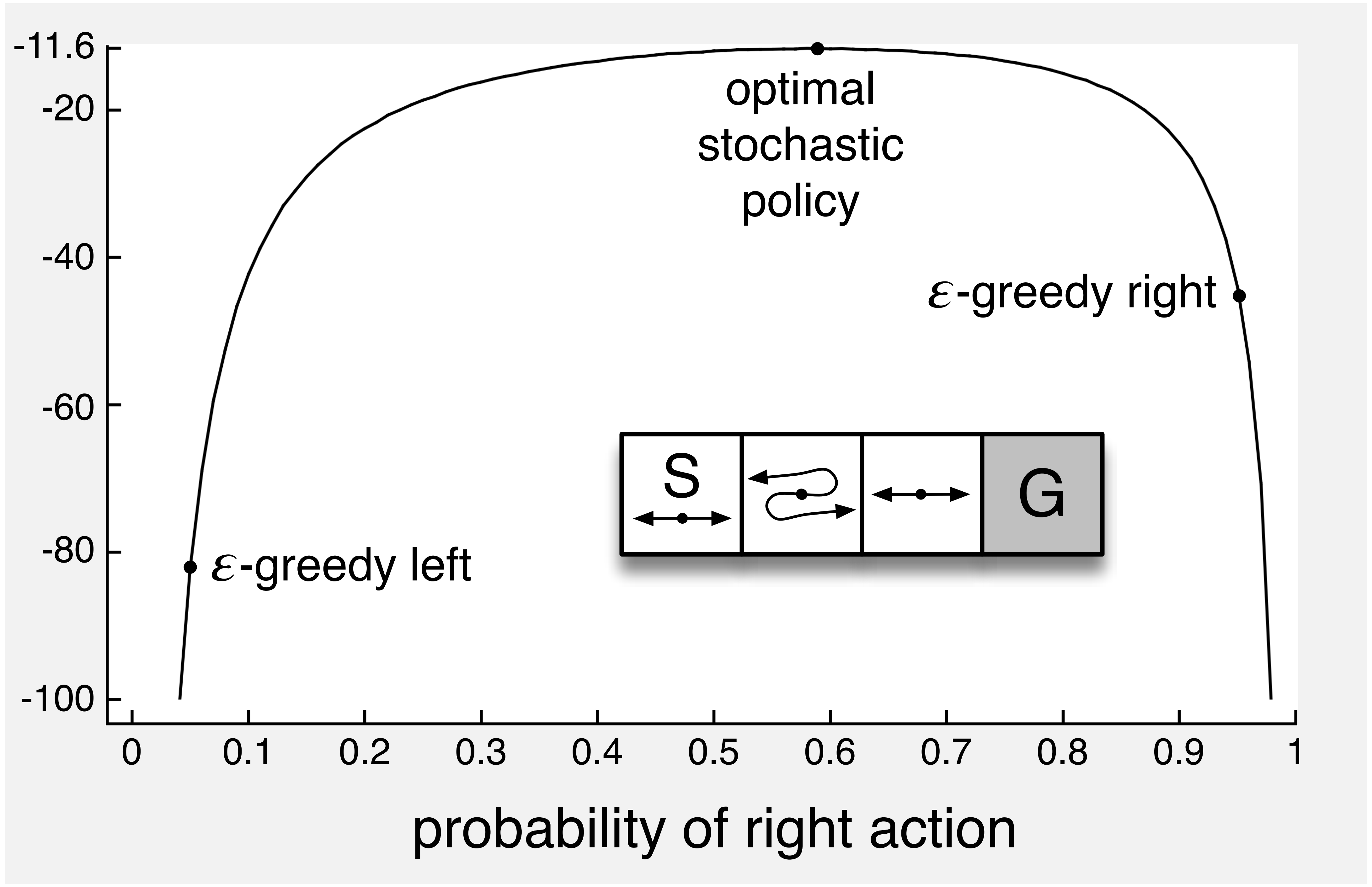
- The **optimal policy** $\pi^*(a \mid s) = \arg\max_a q^*(s, a)$ is typically **deterministic**
- If we run an $\epsilon$-soft policy, we cannot get to an optimal policy
    - **Every action** is played either with probability $\epsilon$ or $(1 - \epsilon)$
- Softmax in **action preference policies** can learn **arbitrary probabilities**, because $h(s, a, \theta)$ is completely **unconstrained**:

$$\pi(a \mid s, \theta) \doteq \frac{e^{h(s,a,\theta)}}{\sum_{a'} e^{h(s,a',\theta)}}$$

- **Question:** How can a softmax in action preferences policy converge to a deterministic policy?
- **Question:** Can you get the same results with $h(s, a, \theta) = \hat{q}(s, a, \theta)$? (**why?**)

# Example: Switcheroo Corridor

- Actions **left** and **right** have usual effect
- Except in one state they are **reversed**!
- Function approximation makes **all** the states look **identical**
- **Optimal policy** is **stochastic**, with $\Pr(\text{right}) \approx 0.59$
- But $\epsilon$-greedy policies can only pick $\Pr(\text{right})$ of $\epsilon$ or $(1-\epsilon)$!



(Image: Sutton & Barto, 2018)

# Parameterized Policies Advantage: Stochastic Actions

- Optimal policies are **deterministic**, but only when there is no **state aggregation**
- When **function approximation** makes states look the same, or when states are **imperfectly observable**, the optimal policy might be an **arbitrary probability distribution**
- Parameterized policies can represent **arbitrary** distributions
  - Although not necessarily arbitrary distributions in **every possible state** (**why not?**)

# Policy Performance

- We choose the policy parameters $\theta$ in order to maximize the **performance** of the policy: $J(\theta)$
- **Question:** What should $J(\theta)$ be in episodic cases?
- **Expected returns** to the policy specified by $\theta$:

$$J(\theta) \doteq \mathbb{E}_{\pi_\theta}\left[G_0\right]$$

- With special **single starting state** $s_0$:

$$J(\theta) \doteq v_{\pi_\theta}(s_0)$$

## Policy Gradient Ascent

1. Want to **maximize performance**: $J(\theta) = v_{\pi_\theta}(s_0)$
2. Gradient gives direction that **J increases**: $\nabla_\theta J(\theta)$
3. Update parameters in **direction of the gradient**:

$$\begin{aligned} \theta_{t+1} &\leftarrow \theta_t + \alpha \nabla_\theta J(\theta_t) \\ &= \theta_t + \alpha \boxed{\nabla_\theta v_{\pi_\theta}(s_0)} \end{aligned}$$

Oops!

# Policy Gradient Theorem

- The **gradient of the policy** $\nabla J(\theta)$ is just the gradient of the value function with respect to the policy $v_{\pi_\theta}(s_0)$
- But we **don't know** the gradient of the **value function**!

**Policy Gradient Theorem:**

$$\nabla J(\theta) \propto \sum_s \underbrace{\mu(s)}_{\text{on-policy stationary distribution}} \sum_a \underbrace{q_\pi(s,a)}_{\text{true action values}} \underbrace{\nabla \pi(a \mid s, \theta)}_{\text{gradient of } \textbf{policy}}$$

# Monte Carlo Policy Gradient

$$\nabla J(\theta) \propto \boxed{\sum_s \mu(s)} \underbrace{\boxed{\sum_a q_\pi(s,a) \nabla \pi(a|s,\theta)}}_{f(s)}$$

$$\sum_s \Pr(s) f(s) = \mathbb{E}[f(S)]$$

$$= \mathbb{E}_\pi \left[ \sum_a q_\pi(S_t,a) \nabla \pi(a|S_t,\theta) \right]$$

$$= \mathbb{E}_\pi \left[ \sum_a q_\pi(S_t,a) \nabla \pi(a|S_t,\theta) \frac{\pi(a|S_t,\theta)}{\pi(a|S_t,\theta)} \right]$$

$$= \mathbb{E}_\pi \left[ \sum_a \pi(a|S_t,\theta) \underbrace{\boxed{q_\pi(S_t,a) \frac{\nabla \pi(a|S_t,\theta)}{\pi(a|S_t,\theta)}}}_{f(a)} \right]$$

$$\sum_a \Pr(a) f(a) = \mathbb{E}[f(A)]$$

$$= \mathbb{E}_\pi \left[ q_\pi(S_t,A_t) \frac{\nabla \pi(A_t|S_t,\theta)}{\pi(A_t|S_t,\theta)} \right]$$

$$\mathbb{E}\left[\mathbb{E}[f(A)]\right] = \mathbb{E}[f(A)]$$

$$= \mathbb{E}_\pi \boxed{\left[ G_t \frac{\nabla \pi(A_t|S_t,\theta)}{\pi(A_t|S_t,\theta)} \right]}$$

# Monte Carlo Policy Gradient Algorithm: REINFORCE

REINFORCE Update: $\theta_{t+1} \leftarrow \theta_t + \alpha G_t \boxed{\frac{\nabla \pi(A_t | S_t, \theta_t)}{\pi(A_t | S_t, \theta_t)}}$

**REINFORCE: Monte-Carlo Policy-Gradient Control (episodic) for $\pi_*$**

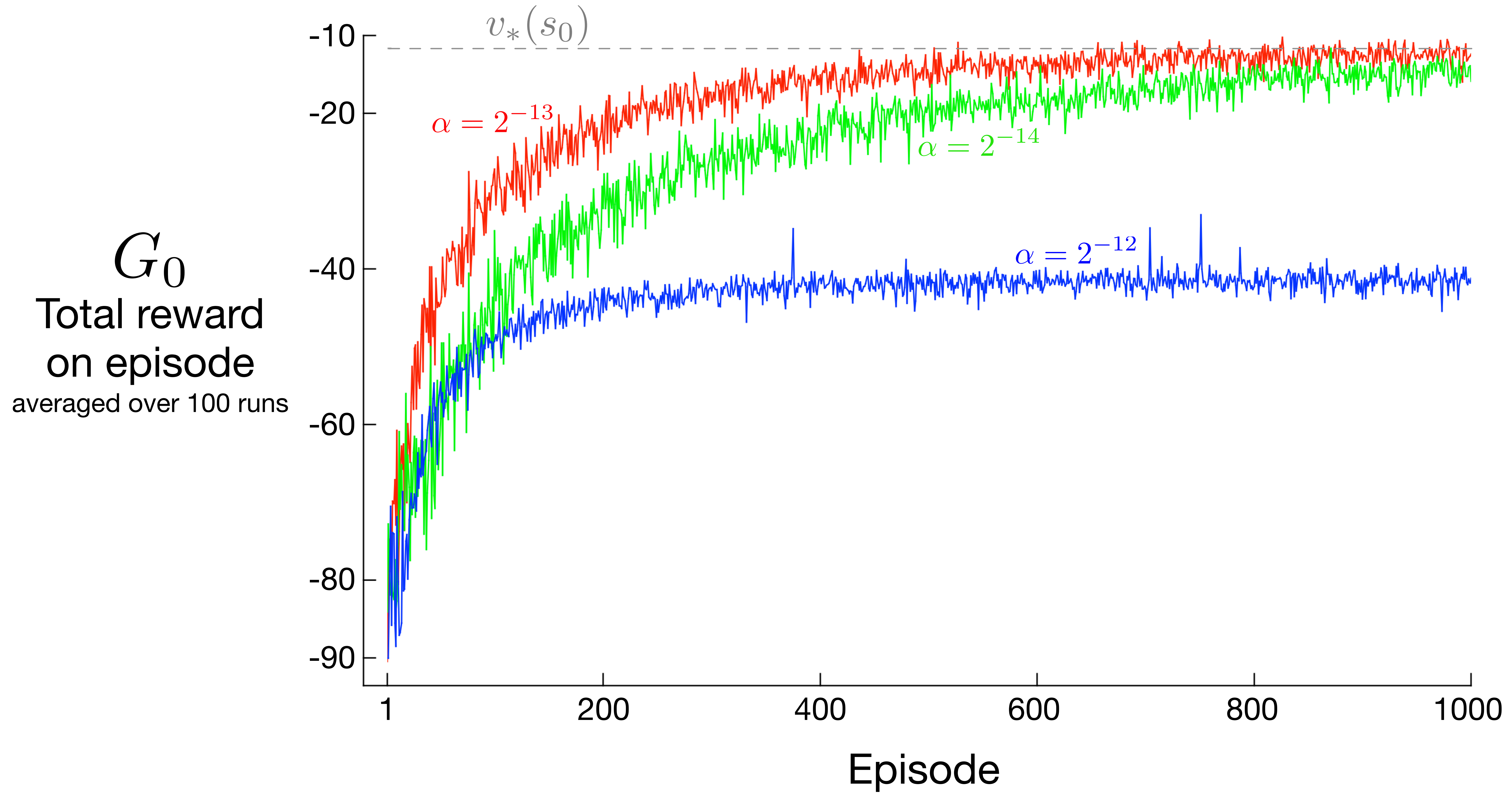
```
Input: a differentiable policy parameterization $\pi(a|s, \boldsymbol{\theta})$
Algorithm parameter: step size $\alpha > 0$
Initialize policy parameter $\boldsymbol{\theta} \in \mathbb{R}^{d'}$ (e.g., to $\mathbf{0}$)

Loop forever (for each episode):
    Generate an episode $S_0, A_0, R_1, \ldots, S_{T-1}, A_{T-1}, R_T$, following $\pi(\cdot|\cdot, \boldsymbol{\theta})$
    Loop for each step of the episode $t = 0, 1, \ldots, T-1$:
        $G \leftarrow \sum_{k=t+1}^{T} \gamma^{k-t-1} R_k$                    $(G_t)$
        $\boldsymbol{\theta} \leftarrow \boldsymbol{\theta} + \alpha \gamma^t G \underbrace{\nabla \ln \pi}_{} (A_t|S_t, \boldsymbol{\theta})$
```

$\frac{\nabla \pi(A_t | S_t, \theta)}{\pi(A_t | S_t, \theta)}$ "eligibility function"

$\left( \nabla \ln x = \frac{\nabla x}{x} \right)$

# REINFORCE Performance in Switcheroo Corridor



(Image: Sutton & Barto, 2018)

# Summary

- It is often impractical to track the estimated value for **every possible state** and/or action
- **Parameterized value function** $\hat{v}(s, \mathbf{w})$ uses weights $\mathbf{w} \in \mathbb{R}^d$ to specify the values of states
    - Weights can be set using **gradient descent** and **semi-gradient descent**
- All our previous control algorithms were **action-value** methods
    1. Approximate the action-value $q^*(s, a)$
    2. Choose maximal-value action at every state
- **Policy gradient** methods:
    1. Represent policies using **parametric policy** $\pi(s \mid a, \theta)$
    2. **Directly optimize** performance $J(\theta)$ by adjusting $\theta$
- **Policy Gradient Theorem** lets us restate $J(\theta)$ in terms of quantities that we **know** ($\nabla \pi$) or can **approximate** ($q_\pi$)
- REINFORCE uses a particular **estimation scheme** for policy gradients