# Calculus Refresher

CMPUT 261: Introduction to Artificial Intelligence

GBC §4.1, 4.3

# Lecture Outline

1. Recap
2. Gradient-based Optimization & Gradients
3. Numerical Issues

*After this lecture, you should be able to:*

- Apply the chain rule of calculus to functions of one or multiple arguments
- Explain the advantages and disadvantages of the method of differences
- Describe the numerical problems with softmax and how to solve them
- Explain why log probabilities are more numerically stable than probabilities

# Loss Minimization

In supervised learning, we choose a **hypothesis** to **minimize** a **loss function**

**Example:** Predict the **temperature**

- *Dataset:* temperatures $y^{(i)}$ from a random sample of days
- *Hypothesis class:* Always predict the ***same value*** $\mu$
- *Loss function:*

$$L(\mu) = \frac{1}{n} \sum_{i=1}^{n} (y^{(i)} - \mu)^2$$

# Optimization

**Optimization:** finding a value of $x$ that **minimizes** $f(x)$

$$x^* = \arg\min_x f(x)$$

- Temperature example: Find $\mu$ that makes $L(\mu)$ small

**Gradient descent:** Iteratively move from current estimate in the direction that makes $f(x)$ **smaller**

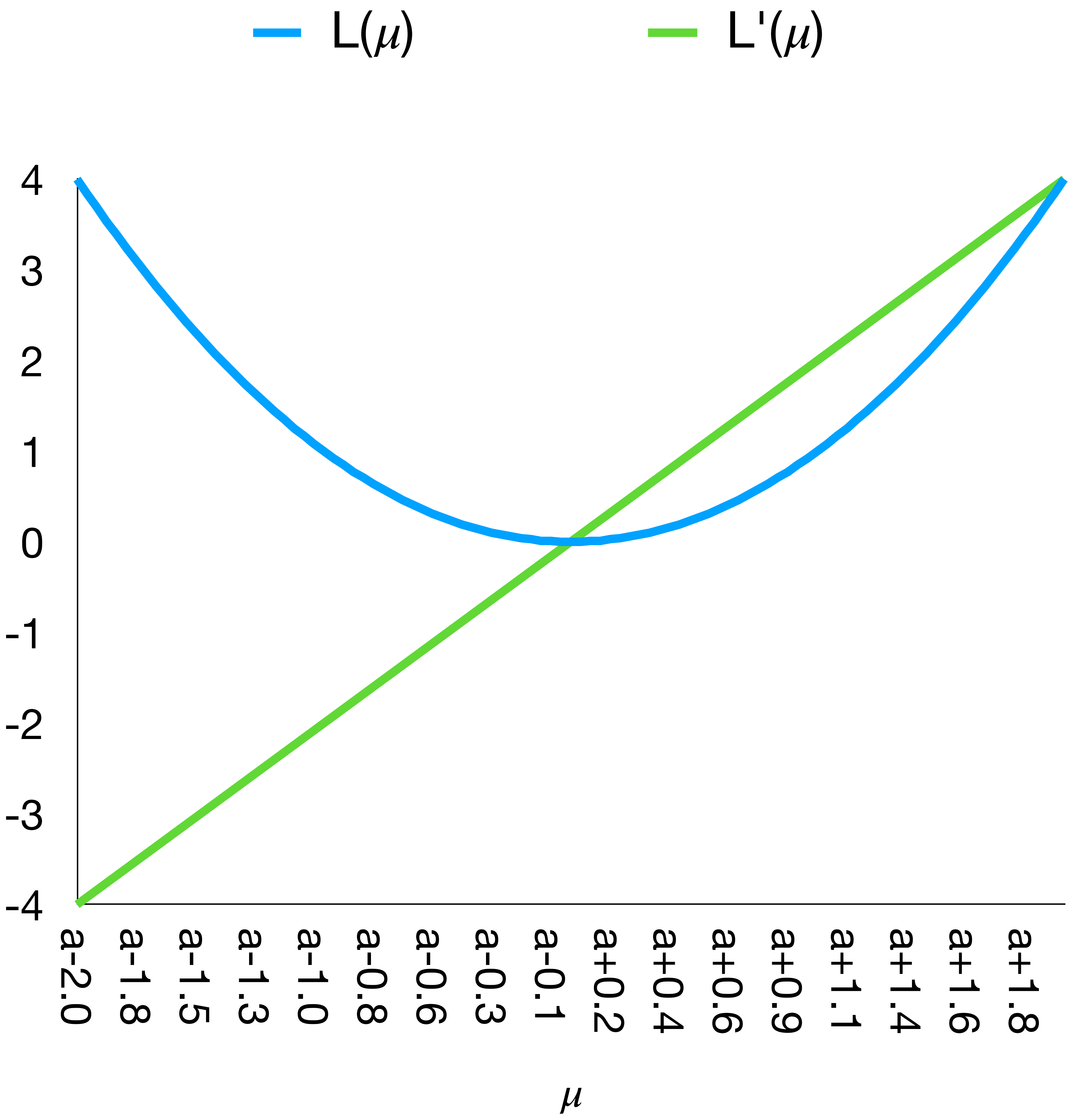
- For **discrete** domains, this is just **hill climbing**:
  Iteratively choose the **neighbour** that has minimum $f(x)$
- For **continuous** domains, neighbourhood is less well-defined

# Derivatives

- The **derivative** $f'(x) = \frac{d}{dx} f(x)$ of a function $f(x)$ is the **slope** of $f$ at point $x$

- When $f'(x) > 0, f$ **increases** with small enough increases in $x$

- When $f'(x) < 0, f$ **decreases** with small enough increases in $x$



# Multiple Inputs

**Example:**
Predict the temperature **based on** pressure and humidity

- *Dataset:*
$$\left(x_1^{(1)}, x_2^{(1)}, y^{(1)}\right), \ldots, \left(x_1^{(m)}, x_2^{(m)}, y^{(m)}\right) = \left\{(\mathbf{x}^{(i)}, y^{(i)}) \mid 1 \leq i \leq m\right\}$$

- *Hypothesis class:* **Linear regression**: $h(\mathbf{x}; \mathbf{w}) = w_0 + w_1 x_1 + w_2 x_2$

- *Loss function:*

$$L(\mathbf{w}) = \frac{1}{n} \sum_{i=1}^{n} \left(y^{(i)} - h(\mathbf{x}^{(i)}; \mathbf{w})\right)^2$$

# Partial Derivatives

**Partial derivatives:** How much does $f(\mathbf{x})$ change when we **only change one** of its inputs $x_i$?

- Can think of this as the derivative of a **conditional** function $g(x_i) = f(x_1, \ldots, \mathbf{x_i}, \ldots, x_n)$:

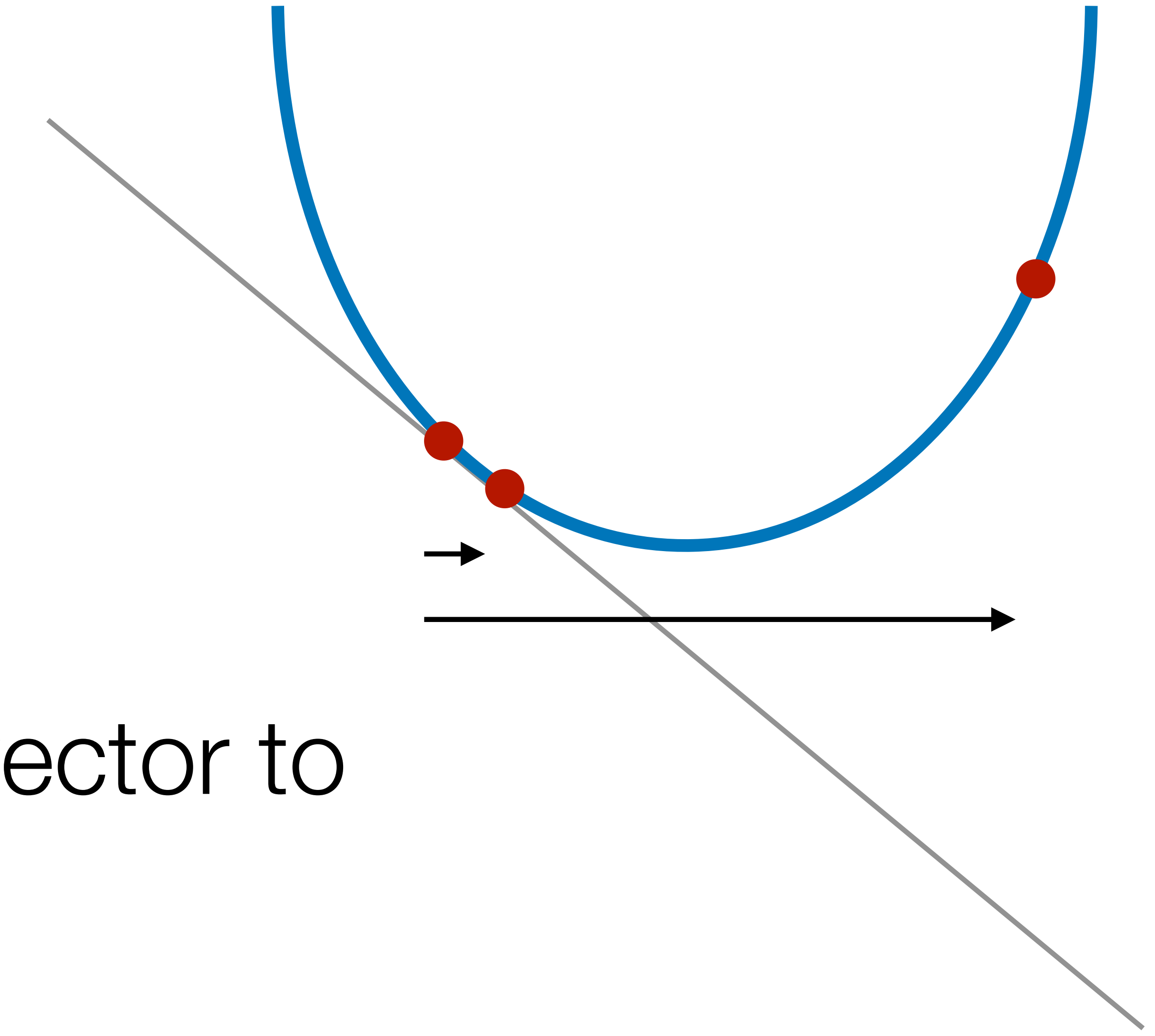
$$\frac{\partial}{\partial x_i} f(\mathbf{x}) = \frac{d}{dx_i} g(x_i).$$

# Gradient

- The **gradient** of a function $f(\mathbf{x})$ is just a **vector** that contains all of its **partial derivatives**:

$$\nabla f(\mathbf{x}) = \begin{bmatrix} \frac{\partial}{\partial x_1} f(\mathbf{x}) \\ \vdots \\ \frac{\partial}{\partial x_n} f(\mathbf{x}) \end{bmatrix}$$

# Gradient Descent



- The gradient of a function tells how to change every element of a vector to **increase** the function
  - If the partial derivative of $x_i$ is positive, increase $x_i$
- **Gradient descent:**
  Iteratively choose new values of x in the (opposite) direction of the gradient:

$$\mathbf{x}^{new} = \mathbf{x}^{old} - \eta \nabla f(\mathbf{x}^{old}) \,.$$

  - This only works for **sufficiently small** changes (**why?**)
  - **Question:** How much should we change $\mathbf{x}^{old}$?

learning rate

# Where Do Gradients Come From?

**Question:** How do we compute the gradients we need for gradient descent?

1. Analytic expressions / direct derivation
2. Method of differences
3. The Chain Rule (of Calculus)

# Analytic Expressions: 1D Derivatives

$$
\begin{aligned}
L(\mu) &= \frac{1}{n}\sum_{i=1}^{n}(y(i)-\mu)^2 \\
&= \frac{1}{n}\sum_{i=1}^{n}\left[y(i)^2 - 2y(i)\mu + \mu^2\right]
\end{aligned}
$$

$$
\frac{d}{d\mu}L(\mu) = \frac{1}{n}\sum_{i=1}^{n}\left[-2y(i) + 2\mu\right]
$$

# Analytic Expressions: Multiple Arguments

To analytically find the **gradient** of a multi-input function, find the **partial derivative** for each of the inputs (and then collect in a vector).

$$
\begin{aligned}
L(\mathbf{w}) &= \frac{1}{n}\sum_{i=1}^{n}\left(y^{(i)} - \mathbf{w}^\top\mathbf{x}^{(i)}\right)^2 \\
&= \frac{1}{n}\sum_{i=1}^{n}\left(y^{(i)} - w_1x_1^{(i)} - w_2x_2^{(i)}\right)^2 \\
&= \frac{1}{n}\sum_{i=1}^{n} w_1^2x_1^{(i)2} + 2w_1w_2x_1^{(i)}x_2^{(i)} - 2w_1x_1^{(i)}y + w_2^2x_2^{(i)2} - 2w_2x_2^{(i)}y + y^2
\end{aligned}
$$

# Analytic Expressions: Multiple Arguments

To analytically find the **gradient** of a multi-input function, find the **partial derivative** for each of the inputs (and then collect in a vector).

$$L(\mathbf{w}) = \frac{1}{n}\sum_{i=1}^{n} w_1^2 x_1^{(i)2} + 2w_1 w_2 x_1^{(i)} x_2^{(i)} - 2w_1 x_1^{(i)} y + w_2^2 x_2^{(i)2} - 2w_2 x_2^{(i)} y + y^2$$

$$\frac{\partial}{\partial w_1} L(w_1, w_2) = \frac{1}{n}\sum_{i=1}^{n} 2w_1 x_1^{(i)2} + 2w_2 x_1^{(i)} x_2^{(i)} - 2x_1^{(i)} y$$

$$\frac{\partial}{\partial w_2} L(w_1, w_2) = \frac{1}{n}\sum_{i=1}^{n} 2w_2 x_2^{(i)2} - 2w_1 x_1^{(i)} x_2^{(i)} + 2x_2^{(i)} y$$

# Analytic Expressions: Multiple Arguments

To analytically find the **gradient** of a multi-input function, find the **partial derivative** for each of the inputs (and then collect in a vector).

$$\nabla L(w_1, w_2) = \begin{bmatrix} \frac{\partial L}{\partial w_1} \\ \frac{\partial L}{\partial w_2} \end{bmatrix} = \begin{bmatrix} \frac{1}{n} \sum_{i=1}^{n} 2w_1 x_1^{(i)2} + 2w_2 x_1^{(i)} x_2^{(i)} - 2x_1^{(i)} y \\ \frac{1}{n} \sum_{i=1}^{n} 2w_2 x_2^{(i)2} - 2w_1 x_1^{(i)} x_2^{(i)} + 2x_2^{(i)} y \end{bmatrix}$$

# Method of Differences

Vector of 0's with a 1 in $i$-th position

e.g., $\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$$\frac{\partial}{\partial w_i} L(\mathbf{w}) \approx L(\mathbf{w} + \epsilon \mathbf{e_i}) - L(\mathbf{w})$$

(for "sufficiently" tiny $\epsilon$)

**Question:** Why would we ever do this?

**Question:** What are the drawbacks?

# Chain Rule (of Calculus): 1D Derivatives

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$$

i.e., $h(x) = f(g(x)) \implies h'(x) = f'(g(x))g'(x)$

- If we know formulas for the derivatives of **components** of a function, then we can build up the derivative of their composition mechanically

- Most prominent example: **Back-propagation** in neural networks

# Chain Rule (of Calculus): Multiple Intermediate Arguments

What if $h(x) = f(g_1(x), g_2(x))$?

$$\frac{dh}{dx} = \frac{\partial f}{\partial g_1}\frac{dg_1}{dx} + \frac{\partial f}{\partial g_2}\frac{dg_2}{dx}$$

**Question:** Why do we **add** the partials via the two arguments?

# Chain Rule (of Calculus): Multiple Arguments

For multiple outputs, things look more complicated, but it's the same idea:

$$h(w_1, w_2) = f(g_1(w_1, w_2), g_2(w_1, w_2))$$

$$\nabla h(w_1, w_2) = \begin{bmatrix} | & | \\ \nabla_{\mathbf{w}} g_1(w_1, w_2) & \nabla_{\mathbf{w}} g_2(w_1, w_2) \\ | & | \end{bmatrix} \nabla_{g(\mathbf{w})} f(g_1(w_1, w_2), g_2(w_1, w_2))$$

$$= \begin{bmatrix} \frac{\partial g_1(w_1, w_2)}{\partial w_1} & \frac{\partial g_2(w_1, w_2)}{\partial w_1} \\ \frac{\partial g_1(w_1, w_2)}{\partial w_2} & \frac{\partial g_2(w_1, w_2)}{\partial w_2} \end{bmatrix} \begin{bmatrix} \frac{\partial f\big(g_1(w_1, w_2), g_2(w_1, w_2)\big)}{\partial g_1(w_1)} \\ \frac{\partial f\big(g_1(w_1, w_2), g_2(w_1, w_2)\big)}{\partial g_2(w_2)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial f\big(g_1(w_1, w_2), g_2(w_1, w_2)\big)}{\partial g_1(w_1)} \frac{\partial g_1(w_1, w_2)}{\partial w_1} + \frac{\partial f\big(g_1(w_1, w_2), g_2(w_1, w_2)\big)}{\partial g_2(w_2)} \frac{\partial g_2(w_1, w_2)}{\partial w_1} \\ \frac{\partial f\big(g_1(w_1, w_2), g_2(w_1, w_2)\big)}{\partial g_1(w_1)} \frac{\partial g_1(w_1, w_2)}{\partial w_2} + \frac{\partial f\big(g_1(w_1, w_2), g_2(w_1, w_2)\big)}{\partial g_2(w_2)} \frac{\partial g_2(w_1, w_2)}{\partial w_2} \end{bmatrix}$$

# Approximating Real Numbers

- Computers store real numbers as **finite number** of bits
- **Problem:** There are an **infinite number** of real numbers in any interval
- Real numbers are encoded as **floating point numbers**:
  - $1.\underbrace{001...011011}_{\textit{significand}} \times 2^{\underbrace{1001..0011}_{\textit{exponent}}}$
  - *Single precision:* 24 bits **significand**, 8 bits **exponent**
  - *Double precision:* 53 bits significand, 11 bits exponent
- **Deep learning** typically uses single precision!

# Underflow

$$1.\underbrace{001\ldots011010}_{significand} \times 2^{\underbrace{1001\ldots0011}_{exponent}}$$

- Positive numbers that are smaller than 1.00...01 × $2^{-1111...1111}$ will be rounded down to **zero**
  - Negative numbers that are bigger than -1.00...01 × $2^{-1111...1111}$ will be **rounded up to zero**
- Sometimes that's okay!  (Almost every number gets rounded)
- Often it's not (**when?**)
  - Denominators: causes divide-by-zero
  - log: returns -inf
  - log(negative): returns nan

# Overflow

$$\underbrace{1.001\ldots011010}_{significand} \times 2^{\overbrace{1001\ldots0011}^{exponent}}$$

- Numbers bigger than 1.111...1111 × $2^{1111}$ will be rounded up to **infinity**
- Numbers smaller than -1.111...1111 × $2^{1111}$ will be rounded down to **negative infinity**
- **exp** is used very frequently
  - Underflows for very negative inputs
  - Overflows for "large" positive inputs
  - **89** counts as "large"

# Addition/Subtraction

$$1.\underbrace{001\ldots011010}_{significand} \times 2^{\overbrace{1001\ldots0011}^{exponent}}$$

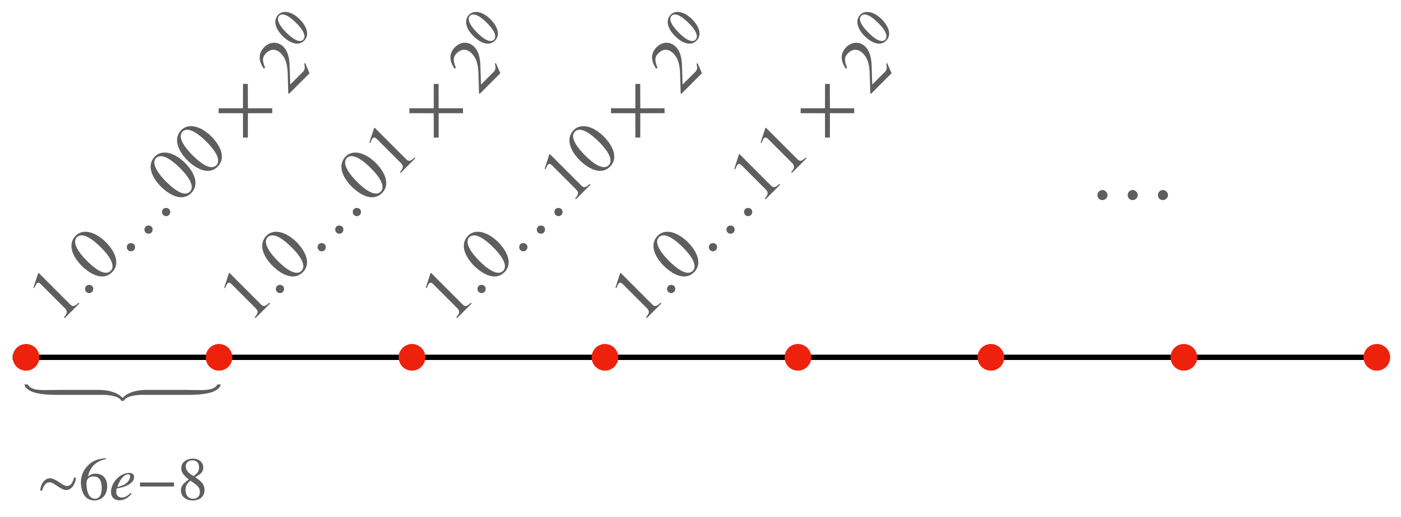
- Adding a small number to a large number can have no effect (**why**?)

**Example:**

```
>>> A = np.array([0., 1e-8])
>>> A = np.array([0., 1e-8]).astype('float32')
>>> A.argmax()
1
>>> (A + 1).argmax()
0

>>> A+1
array([1., 1.], dtype=float32)
```

1e-8 is not the smallest possible float32



$$2^{-24} \approx 5.9 \times 10^{-8}$$

# Softmax

$$softmax(\mathbf{x})_i = \frac{\exp(x_i)}{\sum_{j=1}^{n} \exp(x_j)}$$

- **Softmax** is a very common function
- Used to convert a vector of activations (i.e., numbers) into a **probability distribution**
  - **Question:** Why not normalize them directly without $\exp$?
- But $\exp$ **overflows** very quickly:
  - Solution: $softmax(\mathbf{z})$ where $\mathbf{z} = \mathbf{x} - \max_j x_j$

# Log

- Dataset likelihoods shrink **exponentially** quickly in the **number of datapoints**
- **Example:**
  - Likelihood of a sequence of 5 fair coin tosses = $2^{-5} = 1/32$
  - Likelihood of a sequence of 100 fair coin tosses = $2^{-100}$
- **Solution:** Use log-probabilities instead of probabilities

$$\log(p_1 p_2 p_3 \ldots p_n) = \log p_1 + \ldots + \log p_n$$

- log-prob of 1000 fair coin tosses is $1000 \log 0.5 \approx -693$

# General Solution

- **Question:**
  What is the most general solution to numerical problems?

- ***Standard libraries***

  - PyTorch, Theano, Tensorflow, etc. **detect** common unstable expressions

  - scipy, numpy have stable implementations of many common patterns (e.g., softmax, logsumexp, sigmoid)

# Summary

- **Gradients** are just vectors of **partial derivatives**
  - Gradients point "uphill"
- **Chain Rule of Calculus** lets us compute derivatives of function compositions using derivatives of simpler functions
- **Learning rate** controls how fast we walk uphill
- Deep learning is fraught with **numerical** issues:
  - Underflow, overflow, magnitude mismatches
  - Use **standard implementations** whenever possible