Inference in Belief Networks

CMPUT 261: Introduction to Artificial Intelligence

P&M §8.4

Assignments

- **Assignment #1** is almost marked
  - Grades should be on eClass by tomorrow morning
- **Assignment #2** is now available
  - Due **Feb 16/2023** (one week from this Thursday) at **11:59pm**
  - Total marks=115
    - eClass says 120 because there are **5 possible bonus marks**
    - Don't worry, this will work! :)

Lecture Outline

1. Recap
2. Factors
3. Variable Elimination
4. Further Optimizations

After this lecture, you should be able to:

- encode a factoring of a joint distribution as a collection of factor objects for variable elimination
- define the factor operations used in variable elimination
- describe the high-level steps of variable elimination
- compare efficiency of different variable orderings for variable elimination
- trace an execution of variable elimination

Recap: Belief Networks

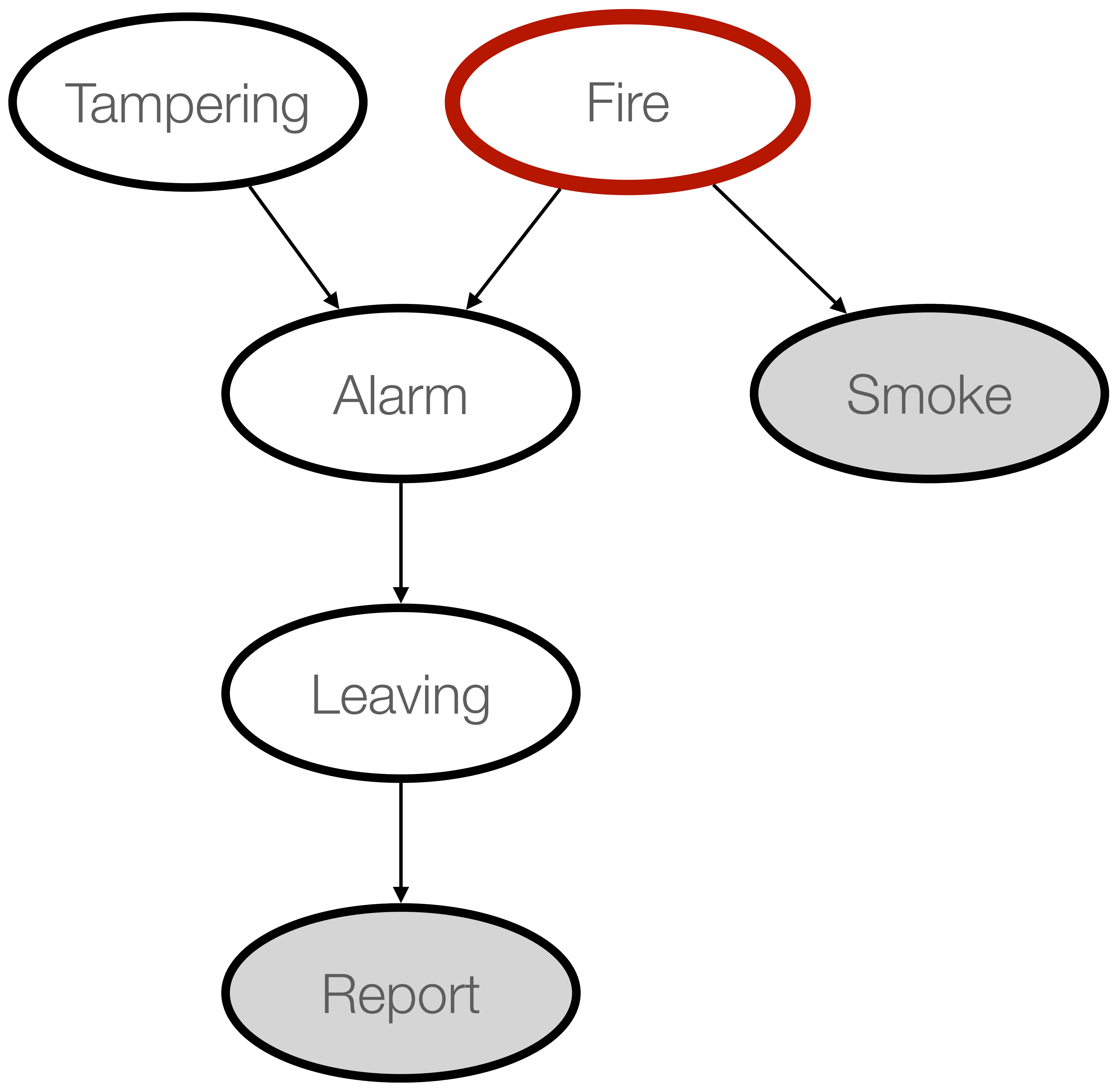
Definition:
A **belief network** (or **Bayesian network**) consists of:

1. A directed acyclic graph, with each node labelled by a **random variable**
2. A **domain** for each random variable
3. A **conditional probability table** for each variable given its **parents**

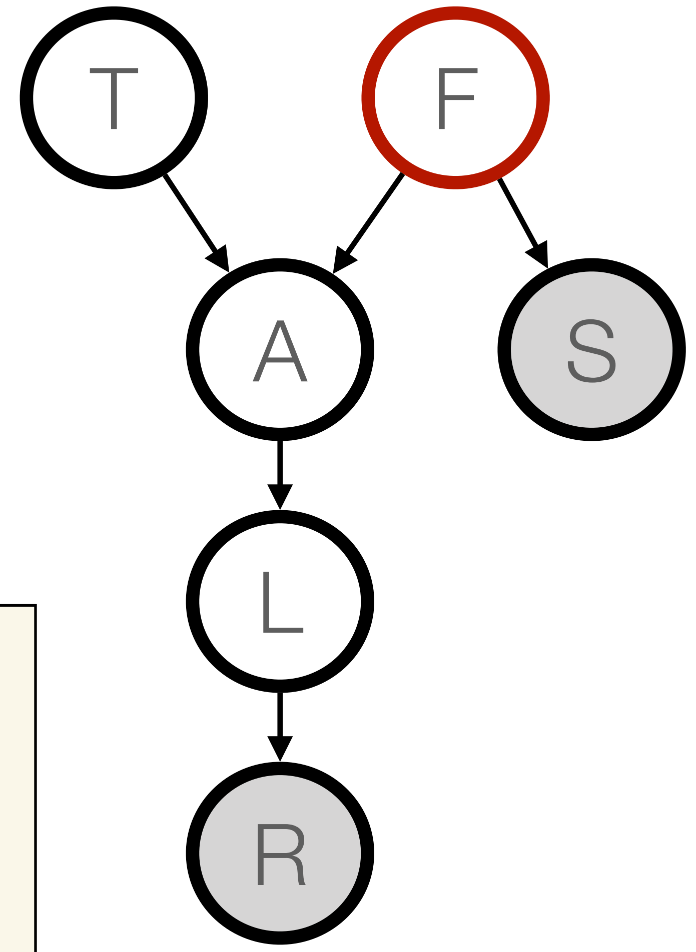
- The graph represents a specific **factorization** of the full **joint distribution**
- **Key Property:**
  Every node is **independent** of its **non-descendants**, **conditional** on its **parents**

Recap: Queries

- The most common task for a belief network is to query **posterior probabilities** given some **observations**
- **Easy cases:**
  - Posteriors of a **single variable** conditional only on **parents**
  - **Joint distributions** of variables early in a **compatible** **variable ordering**
- Typically, the observations have **no straightforward relationship** to the target
- **This lecture:** mechanical procedure for computing **arbitrary queries**



A (Simplistic) Algorithm for Queries



$$P(F, T, A, L, S, R) = P(F)P(T)P(A \mid T, F)P(L \mid A)P(S \mid F)P(R \mid L)$$

Query: $P(F \mid S = 1, R = 1)$

1. **Condition:** $P(F, T, A, L, S = 1, R = 1) = P(F)P(T)P(A \mid T, F)P(S = 1 \mid F)P(L \mid A)P(R = 1 \mid L)$
2. **Normalize:** $P(F, T, A, L \mid S = 1, R = 1) = \dfrac{P(F, T, A, L, S = 1, R = 1)}{\sum_{\substack{f \in \text{dom}(F),\\ t \in \text{dom}(T),\\ a \in \text{dom}(A),\\ l \in \text{dom}(L)}} P(F = f, T = t, A = a, L = l, S = 1, R = 1)}$
3. **Marginalize:** $P(F \mid S = 1, R = 1) = \sum_{\substack{t \in \text{dom}(T),\\ a \in \text{dom}(A),\\ l \in \text{dom}(L)}} P(F, T = t, A = a, L = l \mid S = 1, R = 1)$

Factors

- The **Variable Elimination** algorithm exploits the **factorization** of a joint probability distribution encoded by a belief network in order to answer **queries**
- A **factor** is a function $f(X_1, \ldots, X_k)$ from **random variables** to a **real number**
- **Input:** factors representing the **conditional probability tables** from the belief network

$$P(L \mid A)P(S \mid F)P(A \mid T, F)P(T)P(F)$$

becomes

$$f_1(L, A)f_2(S, F)f_3(A, T, F)f_4(T)f_5(F)$$

- **Output:** A **new factor** encoding the target **posterior distribution**

E.g., $f_{12}(T)$.

Conditional Probabilities as Factors

- A **conditional probability** $P(Y \mid X_1, \ldots, X_n)$ is a factor $f(Y, X_1, \ldots, X_n)$ that obeys the **constraint**:

$$\forall v_1 \in dom(X_1), v_2 \in dom(X_2), \ldots, v_n \in dom(X_n) : \left[\sum_{y \in dom(Y)} f(y, v_1, \ldots, v_n) \right] = 1.$$

- Answer to a query is a factor **constructed** by applying **operations** to the input factors
  - Operations on factors are *not* guaranteed to **maintain** this constraint!
  - Solution: **Don't sweat it**!
  - Operate on **unnormalized probabilities** during the computation
  - **Normalize** at the end of the algorithm to re-impose the constraint

Conditioning

Conditioning is an operation on a **single factor**

- Constructs a **new factor** that returns the values of the original factor with some of its inputs fixed

Definition:

For a factor $f_1(X_1, \ldots, X_k)$, **conditioning on** $X_i = v_i$ yields a new factor

$$f_2(X_1, \ldots X_{i-1}, X_{i+1}, \ldots, X_k) = (f_1)_{X_i = v_i}$$

such that for all values $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k$ in the domain of $X_1, \ldots X_{i-1}, X_{i+1}, \ldots, X_k$,

$$f_2(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k) = f_1(v_1, \ldots, v_{i-1}, \mathbf{v_i}, v_{i+1}, \ldots, v_k).$$

Conditioning Example

$$f_2(A, B) = f_1(A, B, C)_{C=true}$$

f_1

A	B	C	value
F	F	F	0.1
F	F	T	0.88
F	T	F	0.12
F	T	T	0.45
T	F	F	0.7
T	F	T	0.66
T	T	F	0.1
T	T	T	0.25

f_2

A	B	value
F	F	0.88
F	T	0.45
T	F	0.66
T	T	0.25

Multiplication

Multiplication is an operation on **two factors**

- Constructs a new factor that returns the **product** of the rows selected from each factor by its arguments

Definition:

For two factors $f_1(X_1, \ldots, X_j, Y_1, \ldots, Y_k)$ and $f_2(Y_1, \ldots, Y_k, Z_1, \ldots, Z_\ell)$,

multiplication of f_1 and f_2 yields a new factor

$$(f_1 \times f_2) = f_3(X_1, \ldots, X_j, Y_1, \ldots, Y_k, Z_1, \ldots, Z_\ell)$$

such that for all values $x_1, \ldots, x_j, y_1, \ldots, y_k, z_1, \ldots, z_\ell$,

$$f_3(x_1, \ldots, x_j, y_1, \ldots, y_k, z_1, \ldots, z_\ell) = f_1(x_1, \ldots, x_j, y_1, \ldots, y_k) f2(y_1, \ldots, y_k, z_1, \ldots, z_\ell).$$

Multiplication Example

$$f_3(A,B,C) = f_1(A,B) \times f_2(B,C)$$

f_1

A	B	value
F	F	0.1
F	T	0.2
T	F	0.3
T	T	0.4

f_2

B	C	value
F	F	1.0
F	T	0
T	F	0.5
T	T	0.25

f_3

A	B	C	value
F	F	F	0.1
F	F	T	0
F	T	F	0.1
F	T	T	0.05
T	F	F	0.3
T	F	T	0
T	T	F	0.2
T	T	T	0.1

Summing Out

Summing out is an operation on a **single factor**

- Constructs a new factor that returns the **sum over all values** of a random variable of the original factor

Definition:

For a factor $f_1(X_1, \ldots, X_k)$, summing out a variable X_i yields a new factor

$$f_2(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_k) = \left(\sum_{X_i} f_1 \right)$$

such that for all values $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k$ in the domain of $X_1, \ldots X_{i-1}, X_{i+1}, \ldots, X_k$,

$$f_2(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k) = \sum_{\mathbf{v_i} \in dom(X_i)} f_1(v_1, \ldots, v_{i-1}, \mathbf{v_i}, v_{i+1}, \ldots, v_k).$$

Summing Out Example

$$f_2(B) = \sum_A f_1(A, B)$$

f_1

A	B	value
F	F	0.1
F	T	0.2
T	F	0.3
T	T	0.4

f_2

B	value
F	0.4
T	0.6

Variable Elimination

- Given **observations** $Y_1 = v_1, \ldots, Y_k = v_k$ and query variable Q, we want

$$P(Q \mid Y_1 = v_1, \ldots, Y_k = v_k) = \frac{P(Q, Y_1 = v_1, \ldots, Y_k = v_k)}{\sum_{q \in dom(Q)} P(Q = q, Y_1 = v_1, \ldots, Y_k = v_k)}.$$

- Basic idea of variable elimination:

  1. Condition on observations by **conditioning**
  2. Construct joint distribution factor by **multiplication**
  3. Remove unwanted variables (neither query nor observed) by **summing out**
  4. **Normalize** at the end

- Doing these steps in order is **correct** but not **efficient**
- Efficiency comes from **interleaving** the order of operations

Sums of Products

2. Construct joint distribution factor by **multiplication**
3. Remove unwanted variables (neither query nor observed) by **summing out**

The computationally intensive part of variable elimination is computing **sums** of **products**

Example: multiply factors $f_1(Q, A, B, C), f_2(C, D, E)$; sum out A, E

1. $f_3(Q, A, B, C, D, E) = f_1(Q, A, B, C) \times f_2(C, D, E)$: 2^6 multiplications
2. $f_4(Q, B, C, D) = \sum_{A,E} f_3(Q, A, B, C, D, E)$: 3×16 additions

Total: **112** computations

(*) For all numerical examples, we assume binary domains

Efficient Sums of Products

We can reduce the number of computations required by changing their **order**.

$$\sum_A \sum_E f_1(Q, A, B, C) \times f_2(C, D, E)$$

$$= \left(\sum_A f_1(Q, A, B, C) \right) \times \left(\sum_E f_2(C, D, E) \right)$$

1. $f_3(C, D) = \Sigma_E f_2(C, D, E)$: 2^2 additions
2. $f_4(Q, B, C) = \Sigma_A f_1(Q, A, B, C)$: 2^3 additions
3. $f_5(Q, B, C, D) = f_3(Q, B, C) \times f_4(B, C, D)$: 2^4 multiplications

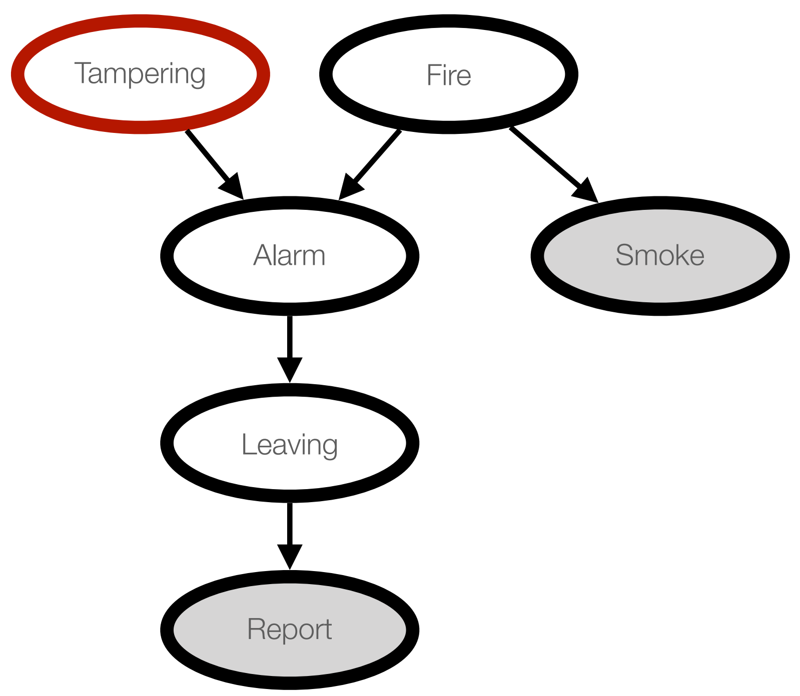
Total: **28** computations

Variable Elimination Algorithm

Input: query variable Q; set of variables Vs; observations O; factors Ps representing conditional probability tables

$Fs := Ps$
for each X in $Vs \setminus \{Q\}$ according to some **elimination ordering**:
    $Rs := \{F \in Fs \mid F \textit{ involves } X\}$
    if $X \in O$:
        for each $F \in Rs$:
            $F' :=F$ **conditioned** on observed value of X
            $Fs := \left(Fs \setminus \{F\}\right) \cup \{F'\}$
    else:
        $T :=$ **product** of factors in Rs
        $N :=$ **sum** X out of T
        $Fs := \left(Fs \setminus Rs\right) \cup \{N\}$
$T :=$ **product** of factors in Fs
$N :=$ **sum** Q out of T
return T/N (i.e., normalize T)

Variable Elimination Example: Conditioning



Query: $P(T \mid S = 1, R = 1)$

Variable ordering: S, R, F, A, L

$$P(T, F, A, S, L, R) = P(T)P(F)P(A \mid T, F)P(S \mid F)P(L \mid A)P(R \mid L)$$

Construct **factors** for each table:

$$\{f_0(T), f_1(F), f_2(T, A, F), f_3(S, F), f_4(L, A), f_5(R, L)\}$$

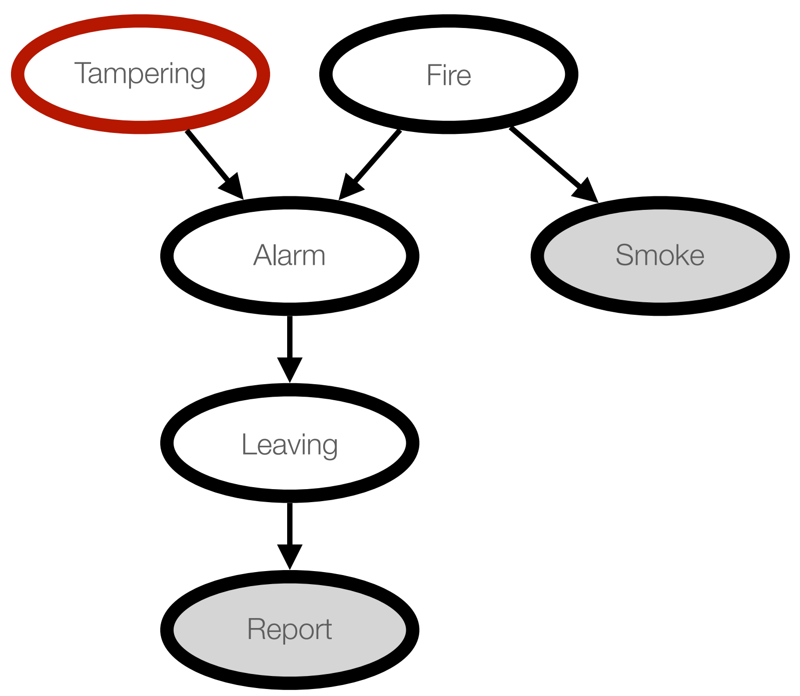
Condition on S: $f_6 = (f_3)_{S=1}$

$$\{f_0(T), f_1(F), f_2(T, A, F), f_6(F), f_4(L, A), f_5(R, L)\}$$

Condition on R: $f_7 = (f_5)_{R=1}$

$$\{f_0(T), f_1(F), f_2(T, A, F), f_6(F), f_4(L, A), f_7(L)\}$$

Variable Elimination Example: Elimination



Query: $P(T \mid S = 1, R = 1)$

Variable ordering: ~~$S, R,$~~ F, A, L

$\{f_0(T), f_1(F), f_2(T, A, F), f_6(F), f_4(L, A), f_7(L)\}$

Sum out F from **product** of f_1, f_2, f_6: $f_8 = \sum_F (f_1 \times f_2 \times f_6)$

$\{f_0(T), f_8(T, A), f_4(L, A), f_7(L)\}$

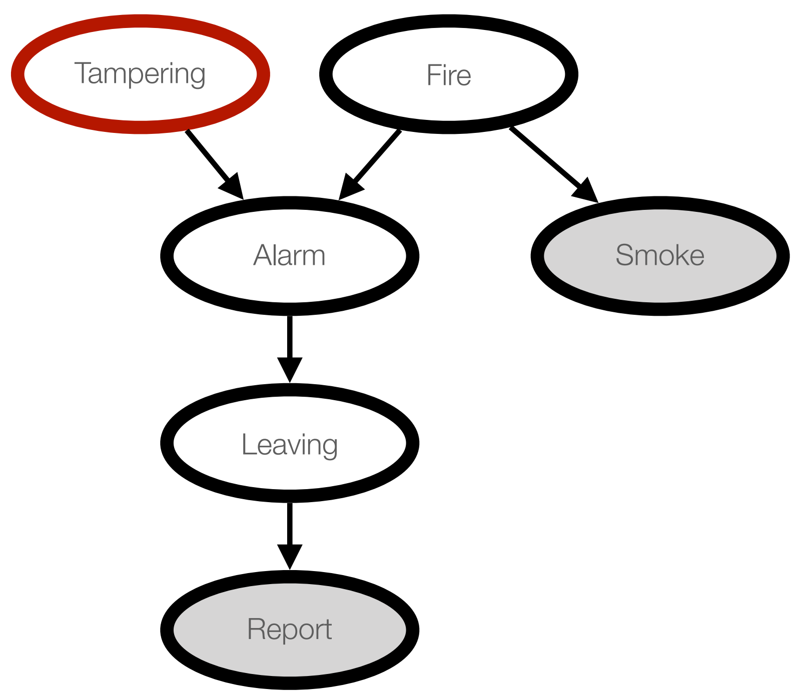
Sum out A from **product** of f_8, f_4: $f_9 = \sum_A (f_8 \times f_4)$

$f_0(T), f_9(T, L), f_7(L)$

Sum out L from **product** of f_9, f_7: $f_{10} = \sum_L (f_9 \times f_7)$

$\{f_0(T), f_{10}(T)\}$

Variable Elimination Example: Normalization



Query: $P(T \mid S = 1, R = 1)$

Variable ordering: ~~S, R, F, A, L~~

$\{f_0(T), f_{10}(T)\}$

Product of remaining factors: $f_{11} = f_0 \times f_{10}$

$\{f_{11}(T)\}$

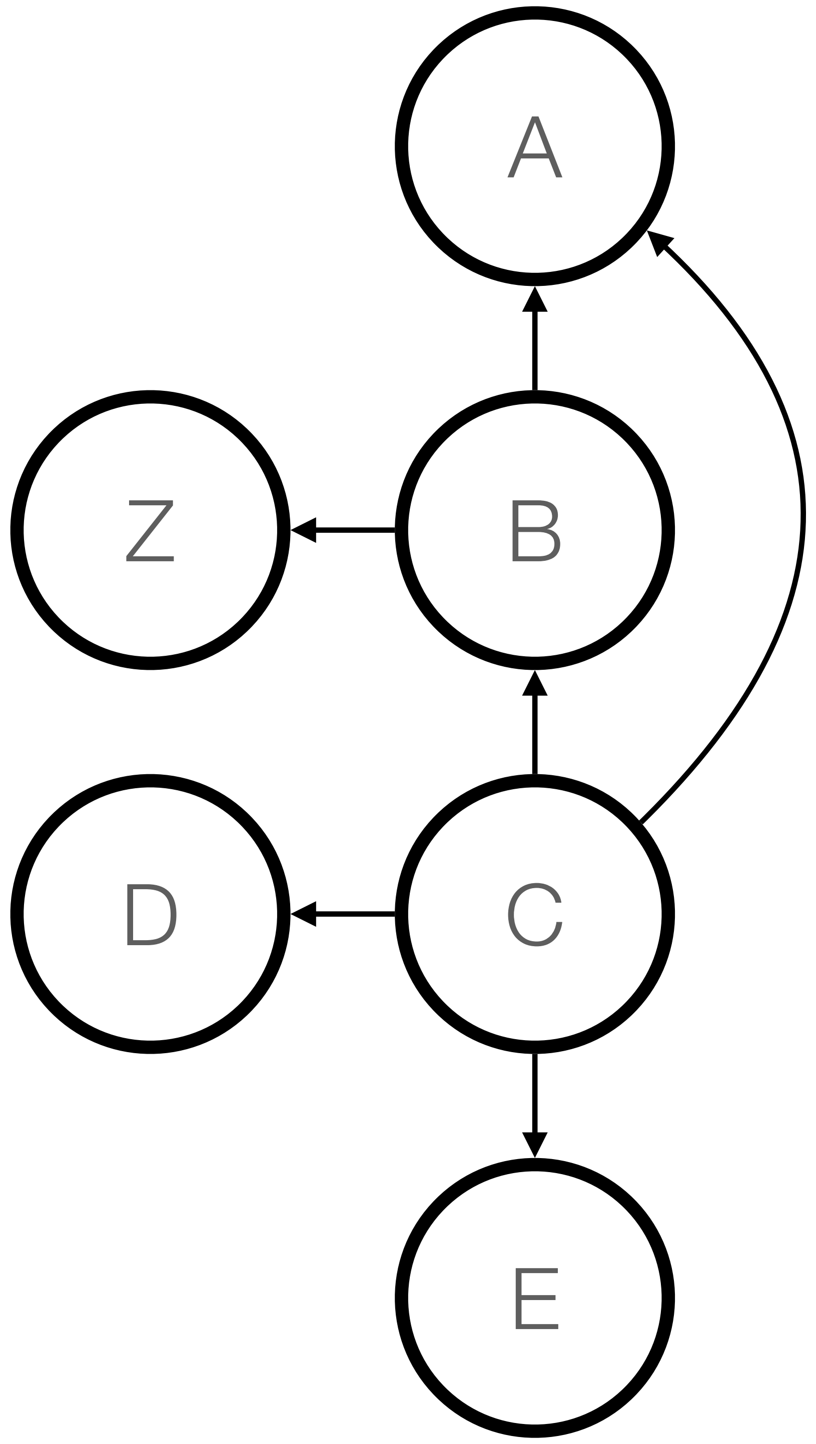
Normalize by division:

$$f_{12}(T) = \frac{f_{11}(T)}{\sum_T f_{11}(T)}$$

Optimizing Elimination Order

- Variable elimination exploits **efficient sums of products** on a **factored joint distribution**
- The **elimination order** of the variables affects the **efficiency** of the algorithm
- Finding an **optimal** elimination ordering is **NP-hard**
- **Heuristics** (rules of thumb) for good orderings:
    - **Observations first:** Condition on all of the observed variables first
    - **Min-factor:** At every stage, select the variable that constructs the **smallest new factor**
    - Problem-specific heuristics

Min-Factor Example



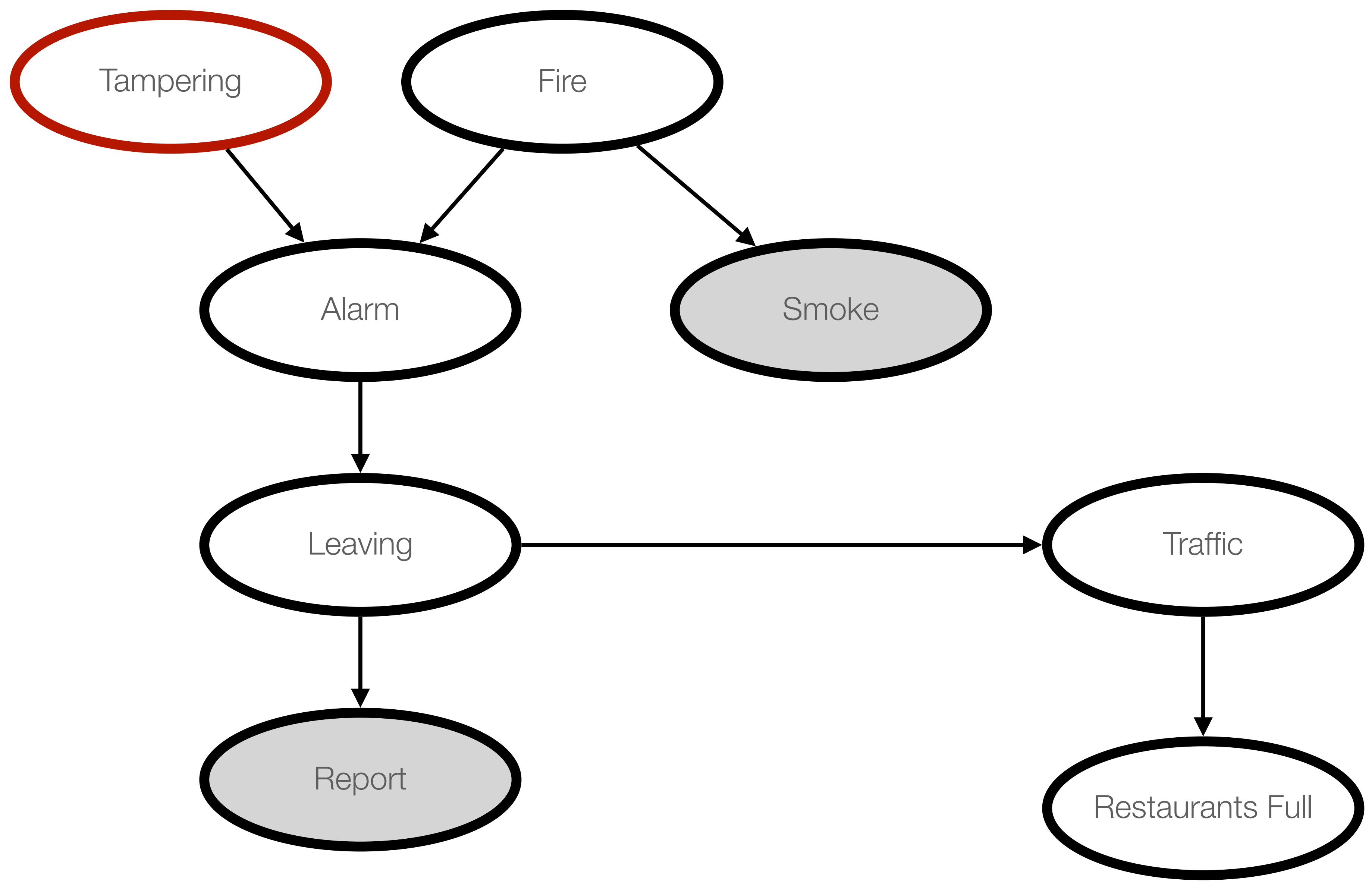
Factors:

$\{f_1(Z,B), f_2(B,C), f_3(C), f_4(D,C), f_5(A,B,C), f_6(E,C)\}$

- Which variable creates the **largest** new factor when it is eliminated?
  - C: Remove $f_2(B,C), f_3(C), f_4(D,C), f_5(A,B,C), f_6(E,C)$, Add $f_7(A,B,D,E)$
- Which variable creates the **smallest** new factor when it is eliminated?
  - Z: Remove $f_1(Z,B)$, add $f_7(B)$
  - (E and A would also work)
  - Number of **rows** is what matters, not number of arguments

Optimization: Pruning

- The structure of the graph can allow us to **drop leaf nodes** that are **neither observed nor queried**
  - Summing them out for **free**
- We can **repeat** this process:



Optimization: Preprocessing

Finally, if we know that we are always going to be observing and/or querying the same variables, we can **preprocess** our graph; e.g.:

1. **Precompute** the **joint distribution** of all the variables we will observe and/or query
2. **Precompute** **conditional distributions** for our exact queries

Summary

- **Variable elimination** is an algorithm for answering **queries** based on a **belief network**
- Operates by using three **operations** on **factors** to reduce graph to a single posterior distribution
    1. Conditioning
    2. Multiplication
    3. Summing out
    4. (Once only): Normalization
- **Distributes** operations more efficiently than taking full product and then summing out
    - **Optimal** order of operations is **NP-hard** to compute
- Additional **optimization** techniques: heuristic ordering, pruning, precomputation