Probability Theory

CMPUT 261: Introduction to Artificial Intelligence

P&M §8.1

Logistics & Assignment #1

- **Assignment #1 was released last week**
  See eClass
  - Due **Tuesday, January 31** at 11:59pm
- Office hours have begun!
  - Not mandatory; for getting help from TAs
  - **There are no labs for this course:** You do not need to show up for your scheduled lab section
- There will be an **example/practice midterm**

Recap: Search

- Agent searches **internal representation** to find solution
- **Fully-observable, deterministic, offline, single-agent** problems
- **Graph search** finds a **sequence of actions** to a goal node
  - Efficiency gains from using **heuristic functions** to encode **domain knowledge**
- **Local search** finds a goal node by repeatedly making **small changes** to the current state
  - **Random steps** and **random restarts** help handle **local optima**, **completeness**

Lecture Outline

1. Recap
2. Uncertainty
3. Probability Semantics
4. Conditional Probability
5. Expected Value

After this lecture, you should be able to:

- Compute joint, marginal, and conditional probabilities
- Compute expected values
- Apply Bayes' rule to compute posterior probabilities
- Apply the Chain rule to compute joint probabilities

Uncertainty

- In search problems, agent has **perfect knowledge** of the world and its dynamics
- In most applications, an agent cannot just **make assumptions** and then act according to those assumptions
- Knowledge is **uncertain**:
  - Must consider **multiple** hypotheses
  - Must **update** beliefs about which hypotheses are likely given **observations**

Example: Wearing a Seatbelt

- An agent has to decide between **three actions**:
    1. Drive without wearing a seatbelt
    2. Drive while wearing a seatbelt
    3. Stay home
- If the agent *knows* that an accident **will** happen, it will just stay home
- If the agent *knows* that an accident **will not** happen, it will not bother to wear a seatbelt!
- Wearing a seatbelt only makes sense because the agent is **uncertain** about whether driving will lead to an accident.

Measuring Uncertainty

- **Probability** is a way of **measuring** uncertainty
- We assign a number between 0 and 1 to **events** (hypotheses):
  - **0** means absolutely certain that statement is **false**
  - **1** means absolutely certain that statement is **true**
  - **Intermediate** values mean more or less certain
- Probability is a measurement of **uncertainty**, **not truth**
  - A statement with probability .75 is not "mostly true"
  - Rather, we **believe** it is more **likely** to be true than not

Subjective vs. Objective: The Frequentist Perspective

- Probabilities can be interpreted
  as **objective** statements about the **world**, or
  as **subjective** statements about an agent's **beliefs**.
- Objective view is called **frequentist:**
  - The probability of an event is the proportion of times it would happen **in the long run** of **repeated experiments**
  - Every event has a single, **true** probability
  - Events that can only happen **once** don't have a well-defined probability

Subjective vs. Objective: The Bayesian Perspective

- Probabilities can be interpreted
  as **objective** statements about the **world**, or
  as **subjective** statements about an agent's **beliefs**.

- Subjective view is called **Bayesian:**

  - The probability of an event is a measure of an agent's **belief** about its likelihood

  - Different agents can legitimately have **different beliefs**, so they can legitimately assign **different probabilities** to the same event

  - There is **only one way** to **update** those beliefs in response to new data

- In this course, we will primarily take the **Bayesian** view

Example: Dice

- Diane rolls a **fair, six-sided die**, and gets the number X
  - **Question:** What is $P(X = 5)$? (the probability that Diane rolled a 5)
- Diane truthfully tells Oliver that she rolled an **odd** number.
  - **Question:** What should **Oliver** believe $P(X = 5)$ is?
- Diane truthfully tells Greta that she rolled a number ≥ 5.
  - **Question:** What should **Greta** believe $P(X = 5)$ is?
- **Question:** What is $P(X = 5)$?

Semantics:
Possible Worlds

- **Random variables** take values from a **domain**.
  We will write them as uppercase letters (e.g., X, Y, D, etc.)
- A **possible world** is a **complete assignment** of values to variables
  We will usually write a single "world" as ω and the set of all possible worlds as Ω
  In this lecture: worlds are **discrete** (i.e., we can take sums)
- A **probability measure** is a function $P : \Omega \to \mathbb{R}$ over **possible worlds** ω satisfying:

1. $$\sum_{\omega \in \Omega} P(\omega) = 1$$
2. $$P(\omega) \geq 0 \;\; \forall \omega \in \Omega$$

Propositions

- A **primitive proposition** is an equality or inequality expression
  E.g., $X = 5$ or $X \geq 4$
- A **proposition** is built up from other propositions using **logical connectives**.
  E.g., $(X = 1 \vee X = 3 \vee X = 5)$
- The **probability** of a proposition is the sum of the probabilities of the **possible worlds in which that proposition is true**:

$$P(\alpha) = \sum_{\omega : \omega \models \alpha} P(\omega)$$

$\omega \models \alpha$ means "α is true in ω"

- Therefore:

$$P(\alpha \vee \beta) \geq P(\alpha)$$

$\alpha \vee \beta$ means "α OR β"

$$P(\alpha \wedge \beta) \leq P(\alpha)$$

$\alpha \wedge \beta$ means "α AND β"

$$P(\neg\alpha) = 1 - P(\alpha)$$

$\neg\alpha$ means "NOT α"

Joint Distributions

- In our dice example, there was a **single** random variable
- We typically want to think about the interactions of **multiple** random variables
- A **joint distribution** assigns a probability to each full assignment of values to variables
  - e.g., $P(X = 1, Y = 5)$. Equivalent to $P(X - 1 \wedge Y = 5)$
  - Can view this as another way of specifying a single **possible world**

Joint Distribution Example

- What might a day be like in Edmonton? Random variables:
  - **Weather**, with domain {clear, snowing}
  - **Temperature**, with domain {mild, cold, very_cold}
- **Joint distribution** P(Weather, Temperature):

Weather	Temperature	P
clear	mild	0.20
clear	cold	0.30
clear	very cold	0.25
snowing	mild	0.05
snowing	cold	0.10
snowing	very cold	0.10

Marginalization

Question:

What is the **marginal distribution** of Weather?

- **Marginalization** is using a joint distribution $P(X_1, \ldots, X_m, \ldots X_n)$ to compute a distribution over a smaller number of variables $P(X_1, \ldots, X_m)$
  - Smaller distribution is called the **marginal distribution** of its variables (e.g., marginal distribution of $X_1, \ldots, X_m$)
- We compute the marginal distribution by summing out the other variables:

$$P(X, Y) = \sum_{w \in \text{dom}(W)} \sum_{z \in \text{dom}(Z)} P(W = w, X, Y, Z = z)$$

Weather	Temperature	P
clear	mild	0.20
clear	cold	0.30
clear	very cold	0.25
snowing	mild	0.05
snowing	cold	0.10
snowing	very cold	0.10

Conditional Probability

- Agents need to be able to **update** their beliefs based on new **observations**
- This process is called **conditioning**
- We write $P(h \mid e)$ to denote "probability of **hypothesis** h given that we have observed **evidence** e"
    - $P(h \mid e)$ is the **probability of h conditional on e**

Semantics of Conditional Probability

- Evidence e lets us **rule out** all of the worlds that are incompatible with e
  - E.g., if I observe that the weather is clear, I should no longer assign **any** probability to the worlds in which it is snowing
  - We need to **normalize** the probabilities of the remaining worlds to ensure that the probabilities of possible worlds sum to 1

$$P(\omega \mid e) = \begin{cases} c \times P(\omega) & \text{if } \omega \models e, \\ 0 & \text{otherwise.} \end{cases}$$

Semantics of Conditional Probability

- Evidence e lets us **rule out** all of the worlds that are incompatible with e
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$$P(\omega \mid e) = \begin{cases} \frac{1}{P(e)} \times P(\omega) & \text{if } \omega \vDash e, \\ 0 & \text{otherwise.} \end{cases}$$

Conditional Probability Example

- My initial marginal belief about the weather was: $P(Weather = snow) = 0.25$
- Suppose I observe that the temperature is **mild**.
  - **Question:** What probability should I **now** assign to $Weather = snow$?

1. **Rule out** incompatible worlds
2. **Normalize** remaining probabilities
3. Result: $P(Weather = snow \mid Temperature = mild) = 0.20$

Weather	P
clear	.20 / (.20 + .05) = **0.8**
snowing	.05 / (.20 + .05) = **0.2**

~~clear~~	~~very cold~~	~~0.25~~
snowing	mild	0.05
~~snowing~~	~~cold~~	~~0.10~~
~~snowing~~	~~very cold~~	~~0.10~~

Chain Rule

Definition: conditional probability

$$P(h \mid e) = \frac{P(h, e)}{P(e)}$$

- We can run this **in reverse** to get

$$P(h, e) = P(h \mid e) \times P(e)$$

Definition: chain rule

$$\begin{aligned} P(\alpha_1, \ldots, \alpha_n) &= P(\alpha_1) \times P(\alpha_2 \mid \alpha_1) \times \cdots \times P(\alpha_n \mid \alpha_1, \ldots, \alpha_{n-1}) \\ &= \Pi_{i=1}^{n} P(\alpha_i \mid \alpha_1, \ldots, \alpha_{i-1}) \end{aligned}$$

Bayes' Rule

- From the **chain rule**, we have

$$P(h, e) = P(h \mid e)P(e)$$
$$= P(e \mid h)P(h)$$

- **Often**, $P(e \mid h)$ is easier to compute than $P(h \mid e)$.

Bayes' Rule:

Likelihood

Posterior

Prior

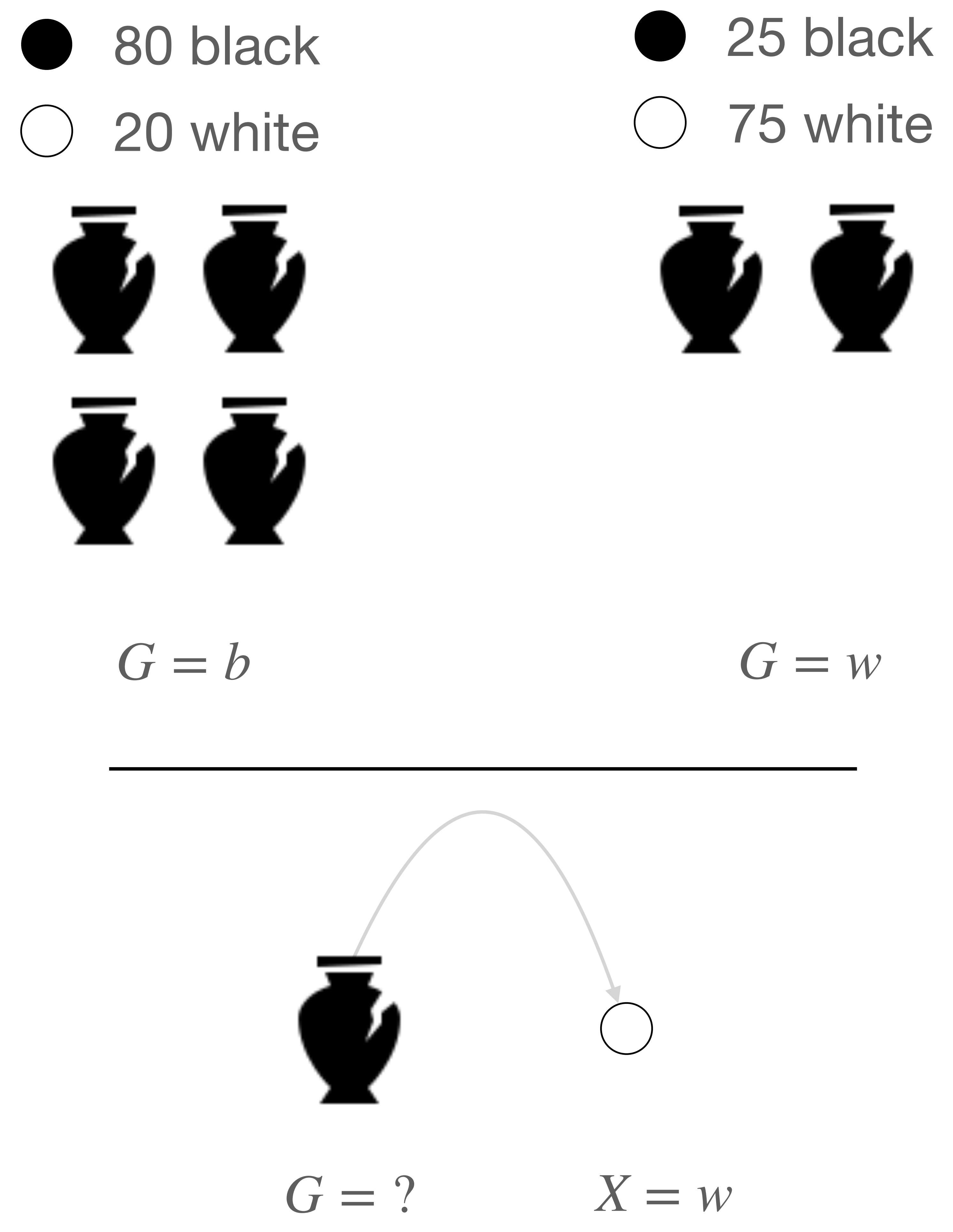
$$\underbrace{\boxed{P(h \mid e)}}_{\text{Posterior}} = \frac{\overbrace{\boxed{P(e \mid h)}}^{\text{Likelihood}} \; \overbrace{\boxed{P(h)}}^{\text{Prior}}}{\underbrace{\boxed{P(e)}}_{\text{Evidence}}}$$

Evidence

Bayes' Rule Example: Urns

- 6 urns with 100 balls each
- Four have 80 black balls, 20 white; the other 2 have 25 black balls, 75 white
- I roll a fair die and choose the urn with the corresponding number
  - **Q:** With what probability are the majority of the balls in the chosen urn white? i.e., $\Pr(G = w)$
- I draw a ball from the urn; it's white! i.e., $X = w$
- **Conditional on that observation**, with what probability are most of the balls in the urn white?

i.e., $\Pr(G = w \mid X = w)$



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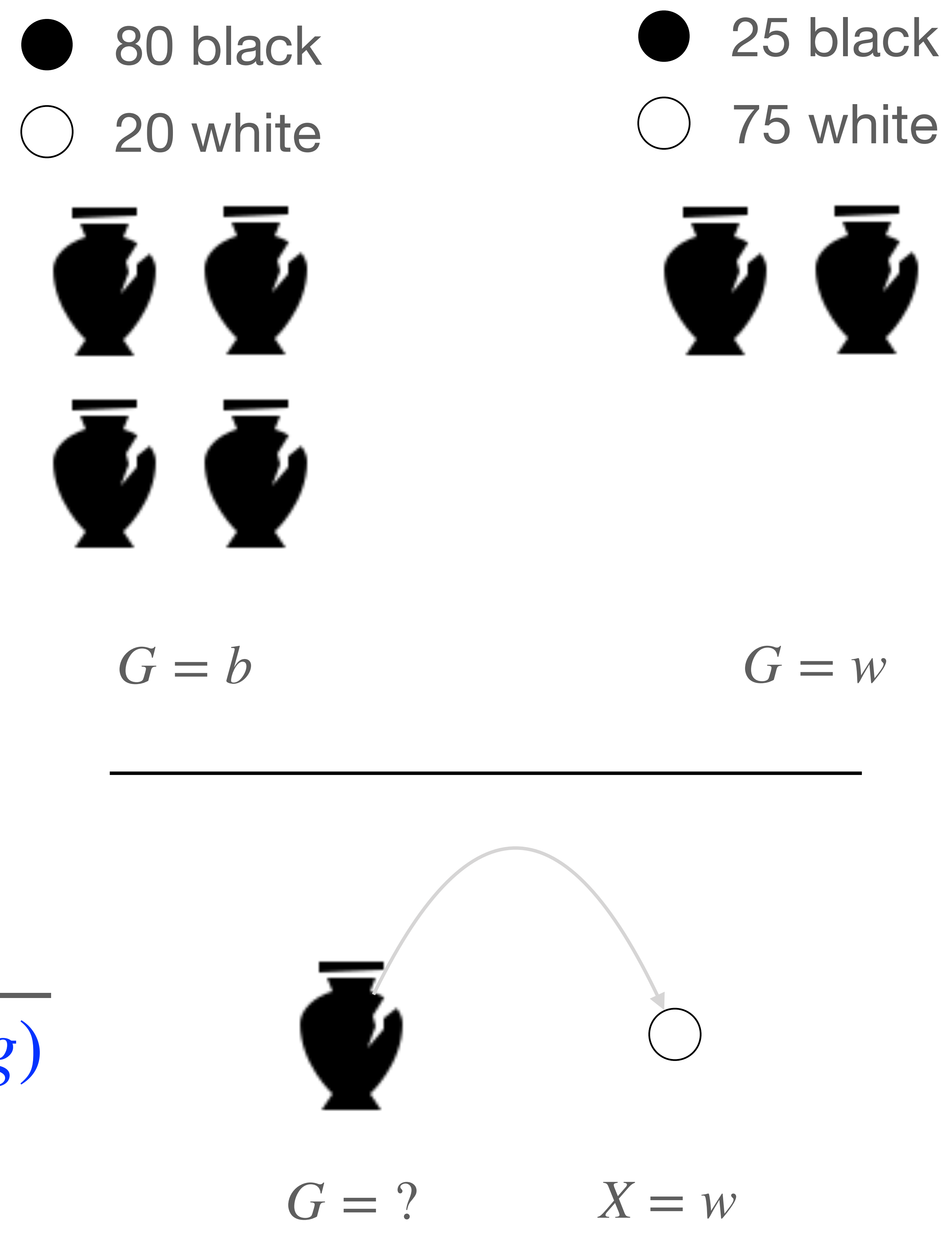
Bayes' Rule Example: Urns

$$\Pr(G = w) = \frac{2}{6}$$

$$\Pr(X = w \mid G = w) = 0.75$$

$$\Pr(G = w \mid X = w) = ?$$

$$\begin{aligned}
\Pr(G = w \mid X = w) &= \frac{\Pr(X = w \mid G = w)\Pr(G = w)}{\Pr(X = w)} \\
&= \frac{\Pr(X = w \mid G = w)\Pr(G = w)}{\sum_{g \in \mathrm{dom}(G)} \Pr(X = w, G = g)} \\
&= \frac{\Pr(X = w \mid G = w)\Pr(G = w)}{\sum_{g \in \mathrm{dom}(G)} \Pr(X = w \mid G = g)\Pr(G = g)} \\
&= \frac{0.75 \times 0.33}{0.75 \times 0.33 + 0.20 \times 0.67}
\end{aligned}$$



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Expected Value

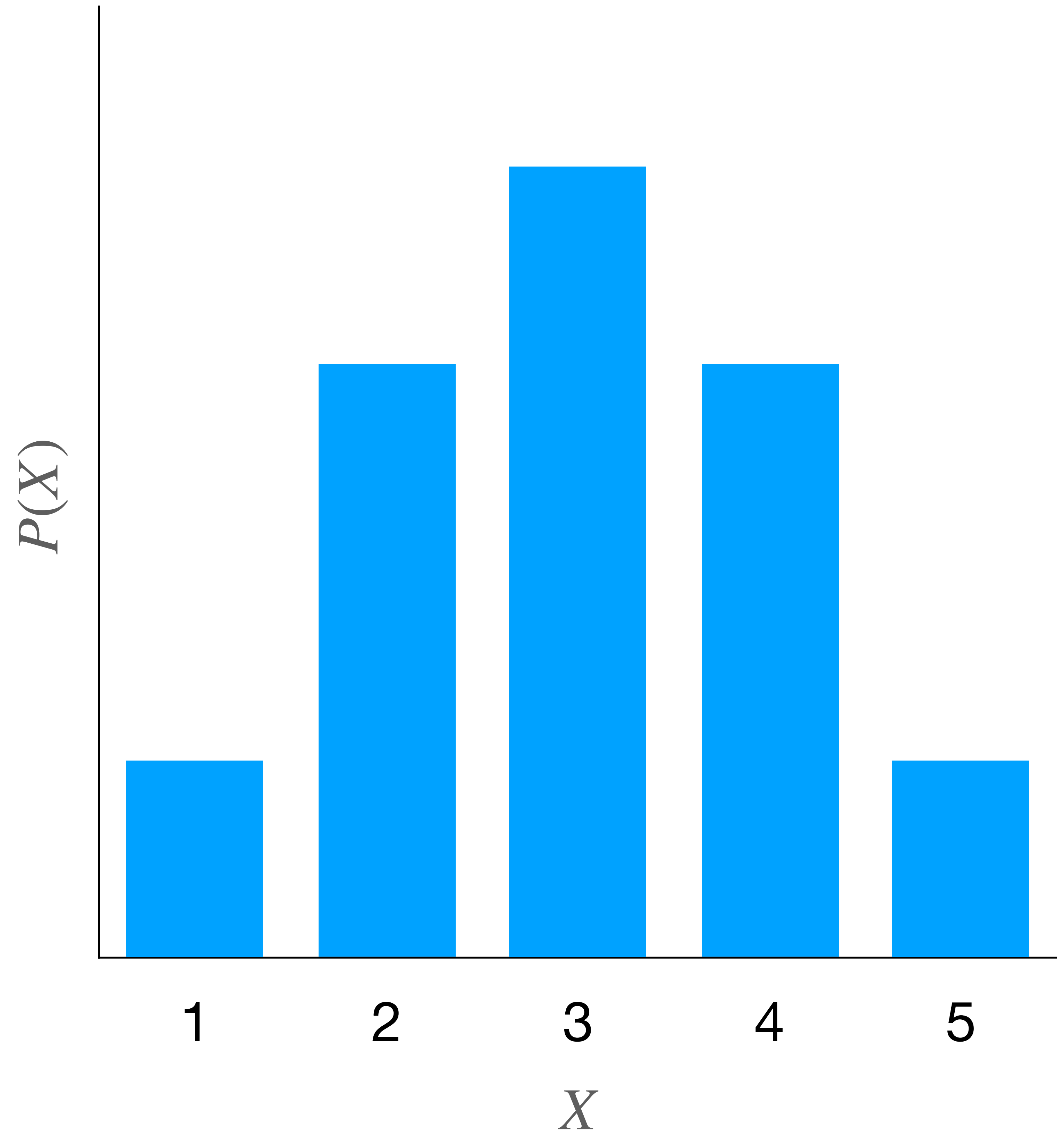
- The **expected value** of a **function** f on a random variable is the weighted **average** of that function over the domain of the random variable, **weighted** by the **probability** of each value:

$$\mathbb{E}\left[f(X)\right] = \sum_{x \in dom(X)} P(X = x) f(x)$$

- The **conditional expected value** of a **function** f is the average value of the function over the domain, weighted by the **conditional probability** of each value:

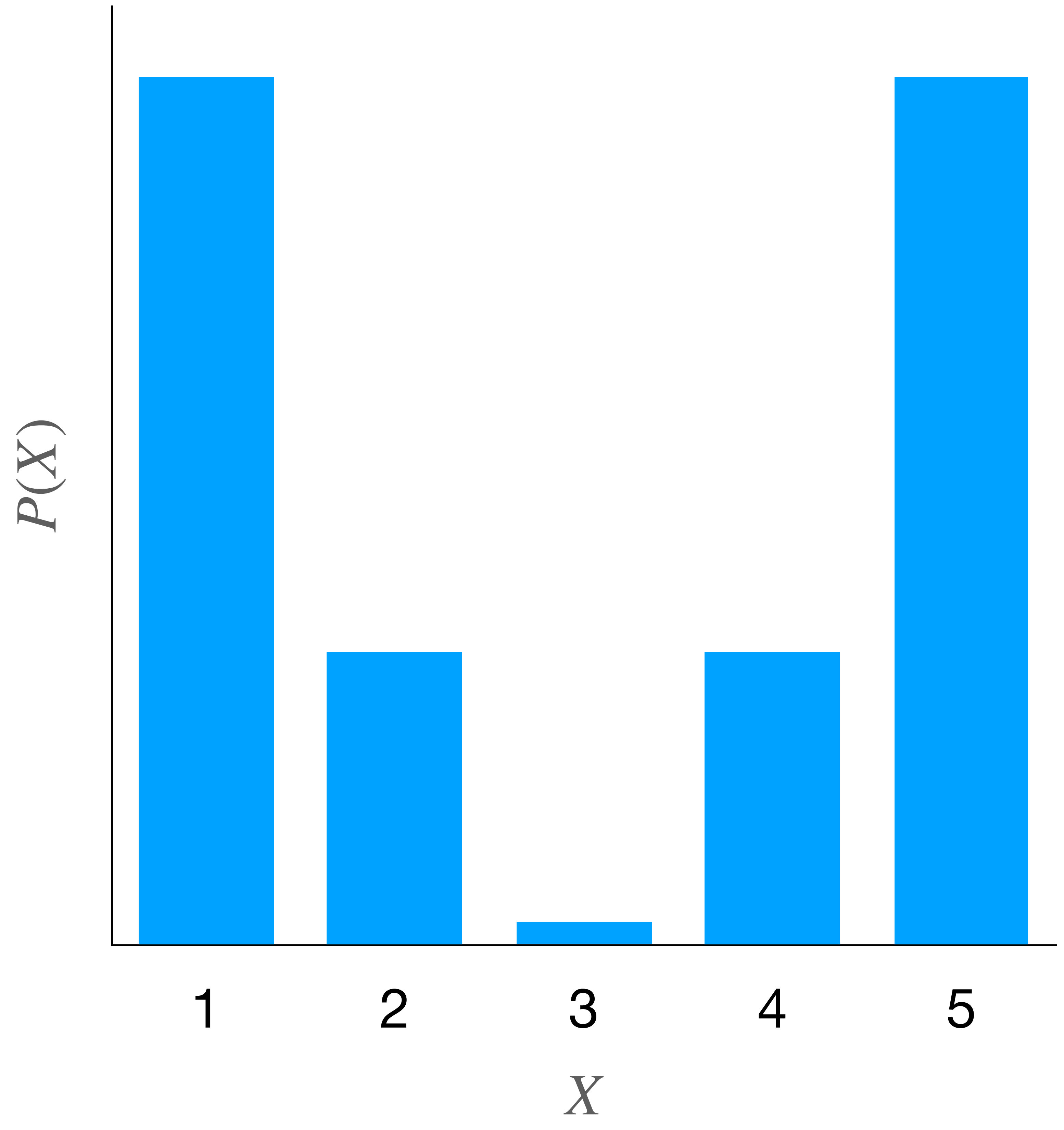
$$\mathbb{E}\left[f(X) \mid Y = y\right] = \sum_{x \in dom(X)} P(X = x \mid Y = y) f(x)$$

Expected Value Examples



$$\mathbb{E}[X] = 3$$

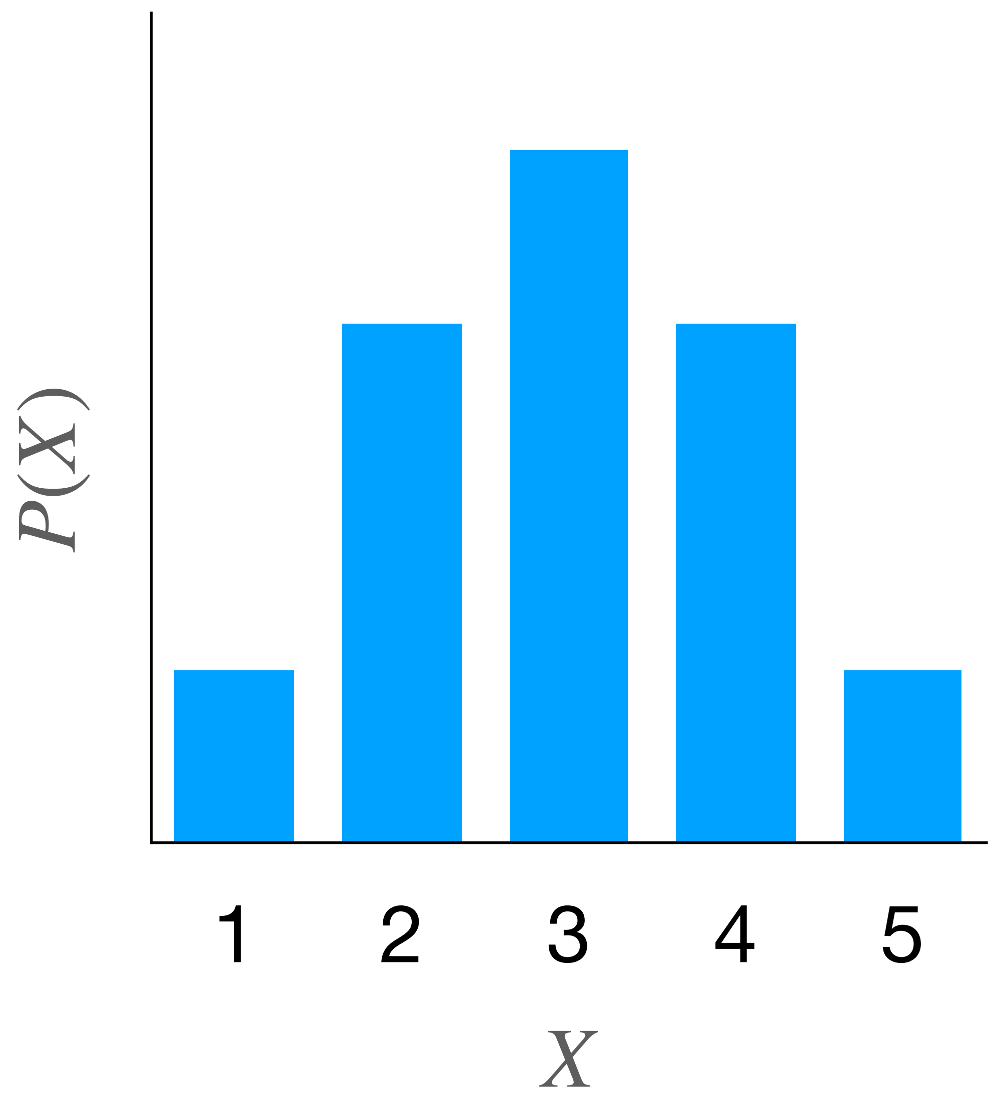
$$\mathbb{E}[X^2] \simeq 10$$

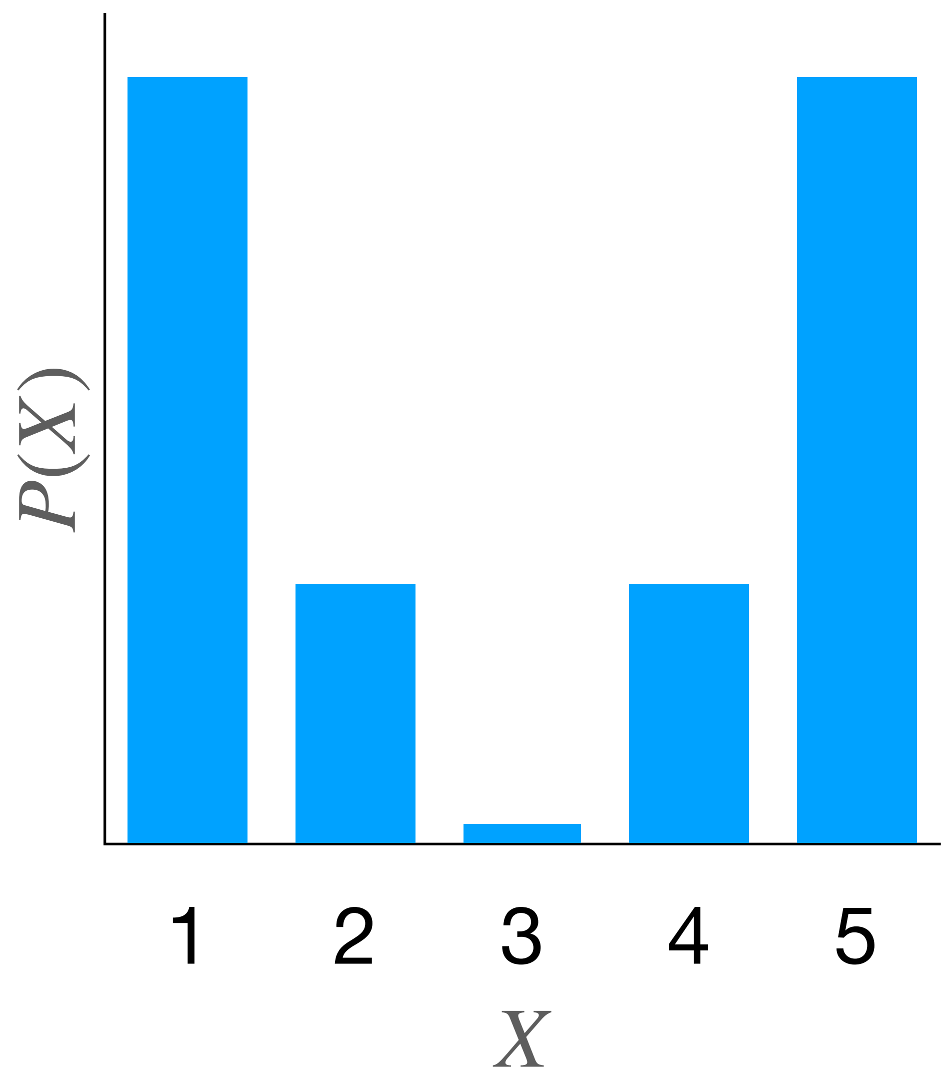


$$\mathbb{E}[X] = 3$$

$$\mathbb{E}[X^2] \simeq 12$$

Expected Value Examples





$$
\begin{aligned}
\mathbb{E}[X] &= \sum_{x \in \mathrm{dom}(X)} x \Pr[X = x] \\
&= \frac{1}{12}1 + \frac{3}{12}2 + \frac{4}{12}3 + \frac{3}{12}4 + \frac{1}{12}5 \\
&= 3
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[X] &= \sum_{x \in \mathrm{dom}(X)} x \Pr[X = x] \\
&= (0.37)1 + (0.125)2 + (0.01)3 + (0.125)4 + (0.37)5 \\
&= 3
\end{aligned}
$$

Summary

- **Probability** is a **numerical** measure of **uncertainty**
- Formal semantics:
  - Weights over **possible worlds** sum to 1
  - Probability of a proposition is **total weight** of **possible worlds** in which that proposition is **true**
- **Conditional probability** updates beliefs based on **evidence**
- **Expected value** of a function is its **probability-weighted average** over possible worlds