Optimality and Dynamic Programming

CMPUT 261: Introduction to Artificial Intelligence

S&B §3.6, §4.0-4.4

Lecture Outline

1. Recap & Logistics
2. Policy Evaluation
3. Optimality
4. Policy Improvement

After this lecture, you should be able to:

- justify why one policy is weakly better than another
- trace an execution of iterative policy evaluation
- state the Policy Improvement Theorem and describe why it is important
- trace an execution of the Value Iteration algorithm

Assignment #4

- Assignment #4 will be released today
    - Due **Tuesday, December 6** at 11:59pm
- Reminder: TAs are available during office hours Mon/Tue/Wed to help

Recap: Value Functions

State-value function

$$v_\pi(s) \doteq \mathbb{E}_\pi[G_t \,|\, S_t = s]$$
$$= \mathbb{E}_\pi\left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \,\middle|\, S_t = s\right]$$

Action-value function

$$q_\pi(s, a) \doteq \mathbb{E}_\pi[G_t \,|\, S_t = s, A_t = a]$$
$$= \mathbb{E}_\pi\left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \,\middle|\, S_t = s, A_t = a\right]$$

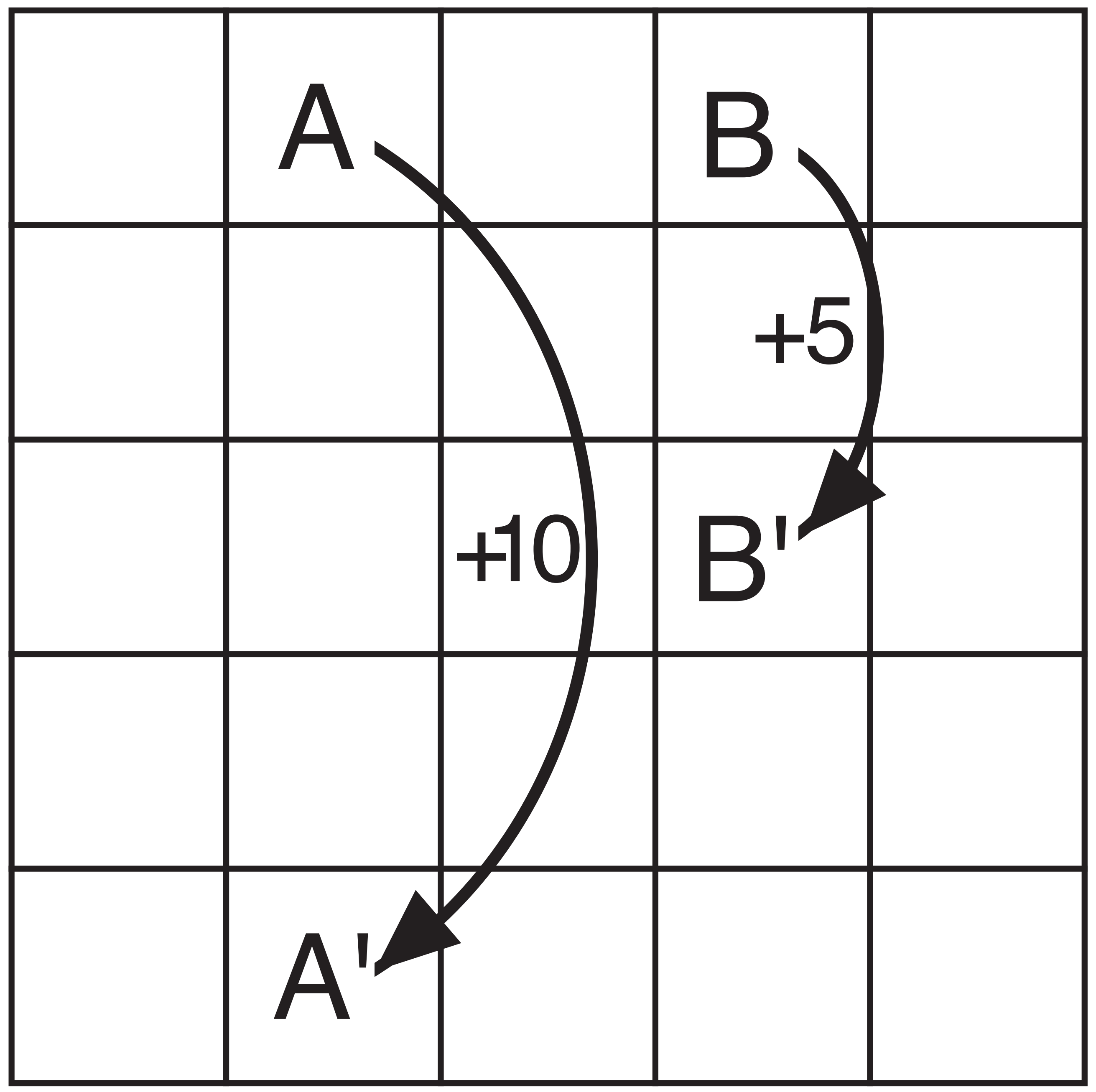
Recap: Bellman Equations

Value functions satisfy a **recursive consistency condition** called the **Bellman equation**:

$$
\begin{aligned}
v_\pi(s) &\doteq \mathbb{E}_\pi[G_t \mid S_t = s] \\
&= \mathbb{E}_\pi[R_{t+1} + \gamma G_{t+1} \mid S_t = s] \\
&= \sum_a \pi(a \mid s) \sum_{s'} \sum_r p(s', r \mid s, a)\Big[r + \gamma \mathbb{E}_\pi[G_{t+1} \mid S_{t+1} = s']\Big] \\
&= \sum_a \pi(a \mid s) \sum_{s',r} p(s', r \mid s, a)\big[r + \gamma v_\pi(s')\big]
\end{aligned}
$$

- v_π is the **unique solution** to π's (state-value) Bellman equation
- There is also a Bellman equation for π's **action-value function**

Recap: GridWorld Example



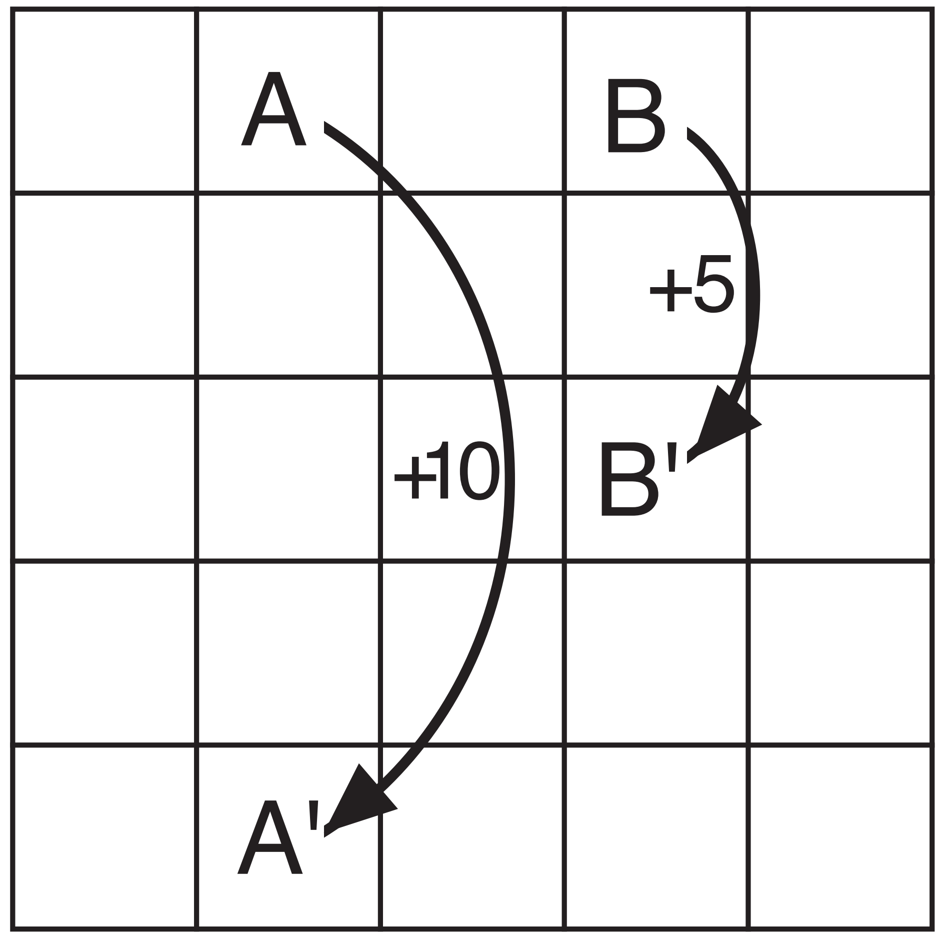
Reward dynamics

3.3	8.8	4.4	5.3	1.5
1.5	3.0	2.3	1.9	0.5
0.1	0.7	0.7	0.4	-0.4
-1.0	-0.4	-0.4	-0.6	-1.2
-1.9	-1.3	-1.2	-1.4	-2.0

State-value function v_π for random policy $\pi(a \mid s) = 0.25$

GridWorld with Bounds Checking

What about a policy where we never try to go over an edge?



Reward dynamics

3.3	8.8	4.4	5.3	1.5
1.5	3.0	2.3	1.9	0.5
0.1	0.7	0.7	0.4	-0.4
-1.0	-0.4	-0.4	-0.6	-1.2
-1.9	-1.3	-1.2	-1.4	-2.0

State-value function v_π for random policy $\pi(a \mid s) = 0.25$

6.7	10.8	6.4	6.7	4.3
4.2	4.7	3.7	3.4	2.8
2.4	2.4	2.1	1.9	1.7
1.5	1.4	1.3	1.2	1.1
1.1	1.0	0.9	0.9	0.9

State-value function v_{π^B} for **bounded** random policy π^B

Policy Evaluation

Question: How can we **compute** v_π?

1. We know that v_π is the unique solution to the Bellman equations, so we could just **solve them** (treating $v_\pi(s_1), \ldots, v_\pi(s_{|\mathcal{S}|})$ as variables)
   - but that is tedious and annoying and slow
     (it's a system of $|\mathcal{S}|$ linear equations in $|\mathcal{S}|$ unknowns)
   - Also requires a complete model of the dynamics
2. **Iterative policy evaluation**
   - Takes advantage of the recursive formulation

Iterative Policy Evaluation

- Iterative policy evaluation uses the Bellman equation as an **update rule**:

$$\begin{aligned} v_{k+1}(s) &\doteq \mathbb{E}_\pi[R_{t+1} + \gamma v_k(S_{t+1} \mid S_t = s] \\ &= \sum_a \pi(a \mid s) \sum_{s',r} p(s', r \mid s, a)\left[r + \gamma v_k(s')\right] \end{aligned}$$

- v_π is a **fixed point** of this update, by definition
- Furthermore, starting from an **arbitrary** v_0, the sequence $\{v_k\}$ will **converge** to v_π as $k \to \infty$

In-Place Iterative Policy Evaluation

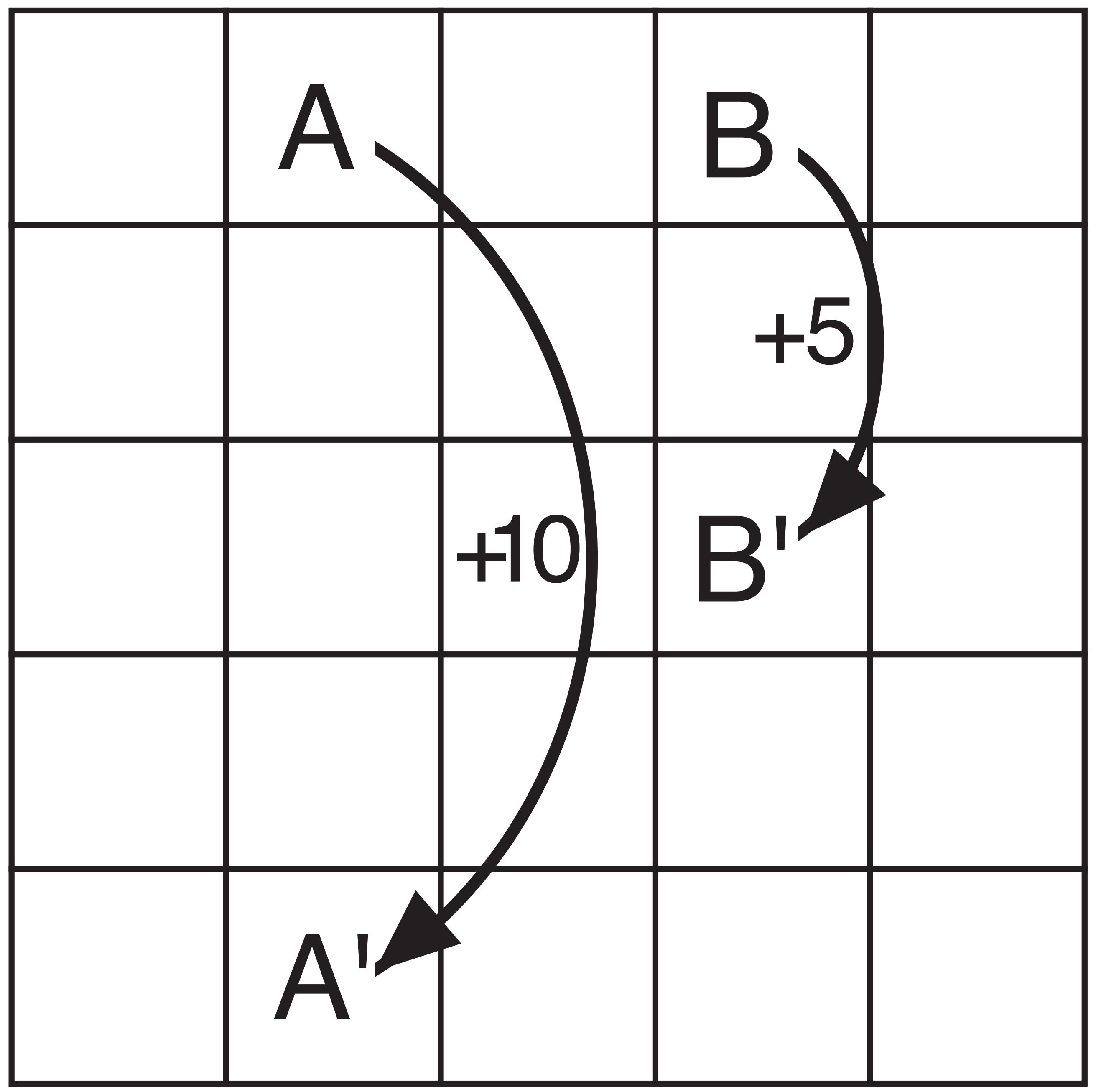
Iterative Policy Evaluation, for estimating $V \approx v_\pi$

Input π, the policy to be evaluated
Algorithm parameter: a small threshold $\theta > 0$ determining accuracy of estimation
Initialize $V(s)$, for all $s \in \mathcal{S}^+$, arbitrarily except that $V(terminal) = 0$

Loop:
    $\Delta \leftarrow 0$
    Loop for each $s \in \mathcal{S}$:
        $v \leftarrow V(s)$
        $V(s) \leftarrow \sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a)\big[r + \gamma V(s')\big]$
        $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$

- The updates are in-place: we use new values for $V(s)$ immediately instead of waiting for the current sweep to complete (**why?**)
- These are expected updates: Based on a weighted average (expectation) of all possible next states (**instead of what?**)

Iterative Policy Evaluation



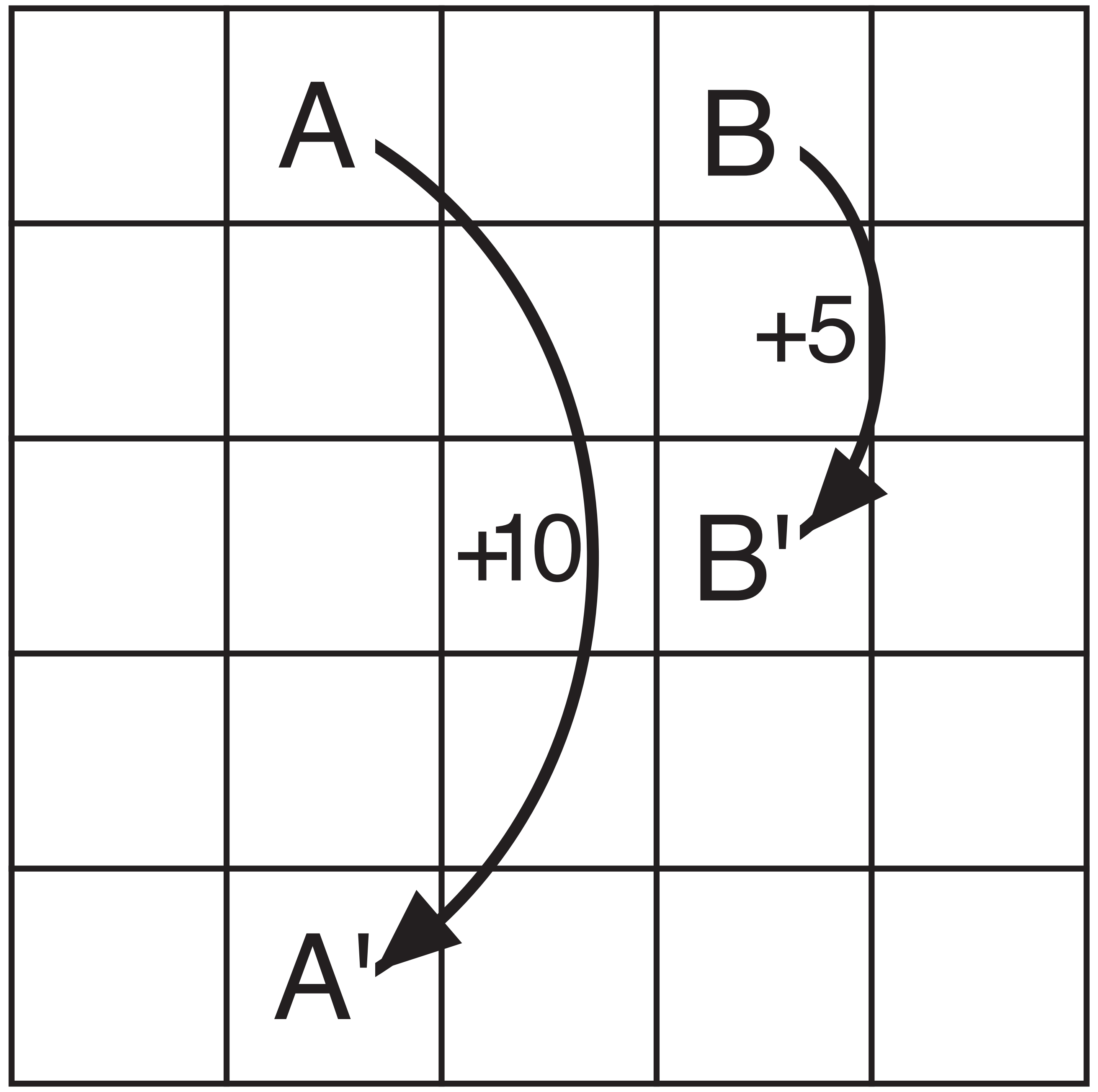
Reward dynamics

0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0

V at $k = 0$

Iterative Policy Evaluation

$$
\begin{aligned}
V(s_{1,1}) &= \pi(\mathsf{n})[-1+\gamma V(s_{1,1})] + \pi(\mathsf{w})[-1+\gamma V(s_{1,1})] + \\
&\quad \pi(\mathsf{s})[0+\gamma V(s_{1,2})] + \pi(\mathsf{e})[0+\gamma V(s_{2,1})] \\
&= 0.25(-1) + 0.25(-1) + 0.25(0) + 0.25(0)
\end{aligned}
$$



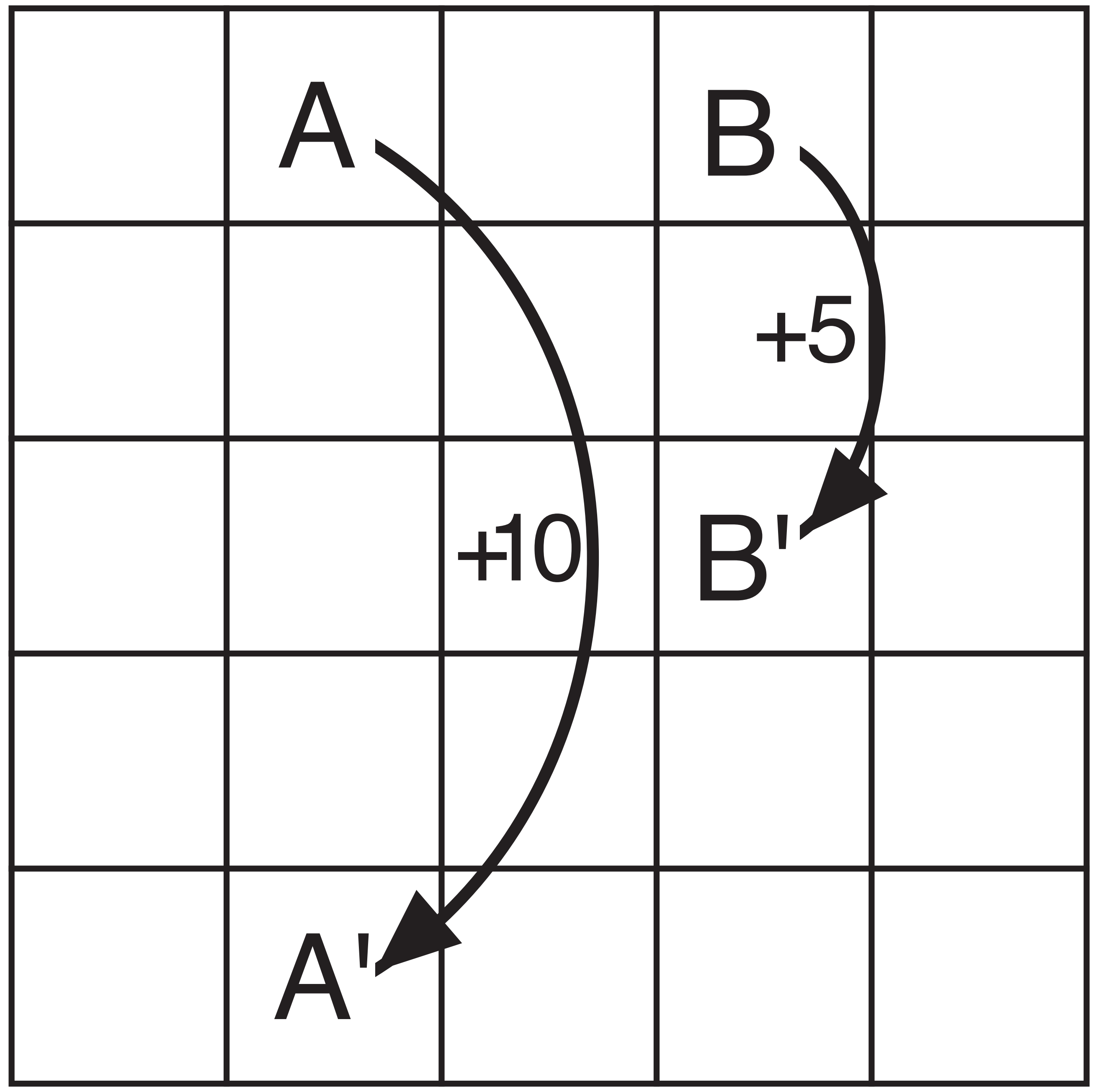
Reward dynamics

-0.5	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0

V at $k = 0$

Iterative Policy Evaluation

$$
\begin{aligned}
V(s_{1,2}) &= \pi(\mathrm{n})[-1+\gamma\mathbf{V(s_{2,5})}] + \pi(\mathrm{w})[-1+\gamma\mathbf{V(s_{2,5})}] + \\
&\quad \pi(\mathrm{s})[0+\gamma\mathbf{V(s_{2,5})}] + \pi(\mathrm{e})[0+\gamma\mathbf{V(s_{2,5})}] \\
&= 0.25[10+0.9(\mathbf{0})] + 0.25[10+0.9(\mathbf{0})] + \\
&\quad 0.25[10+0.9(\mathbf{0})] + 0.25[10+0.9(\mathbf{0})]
\end{aligned}
$$



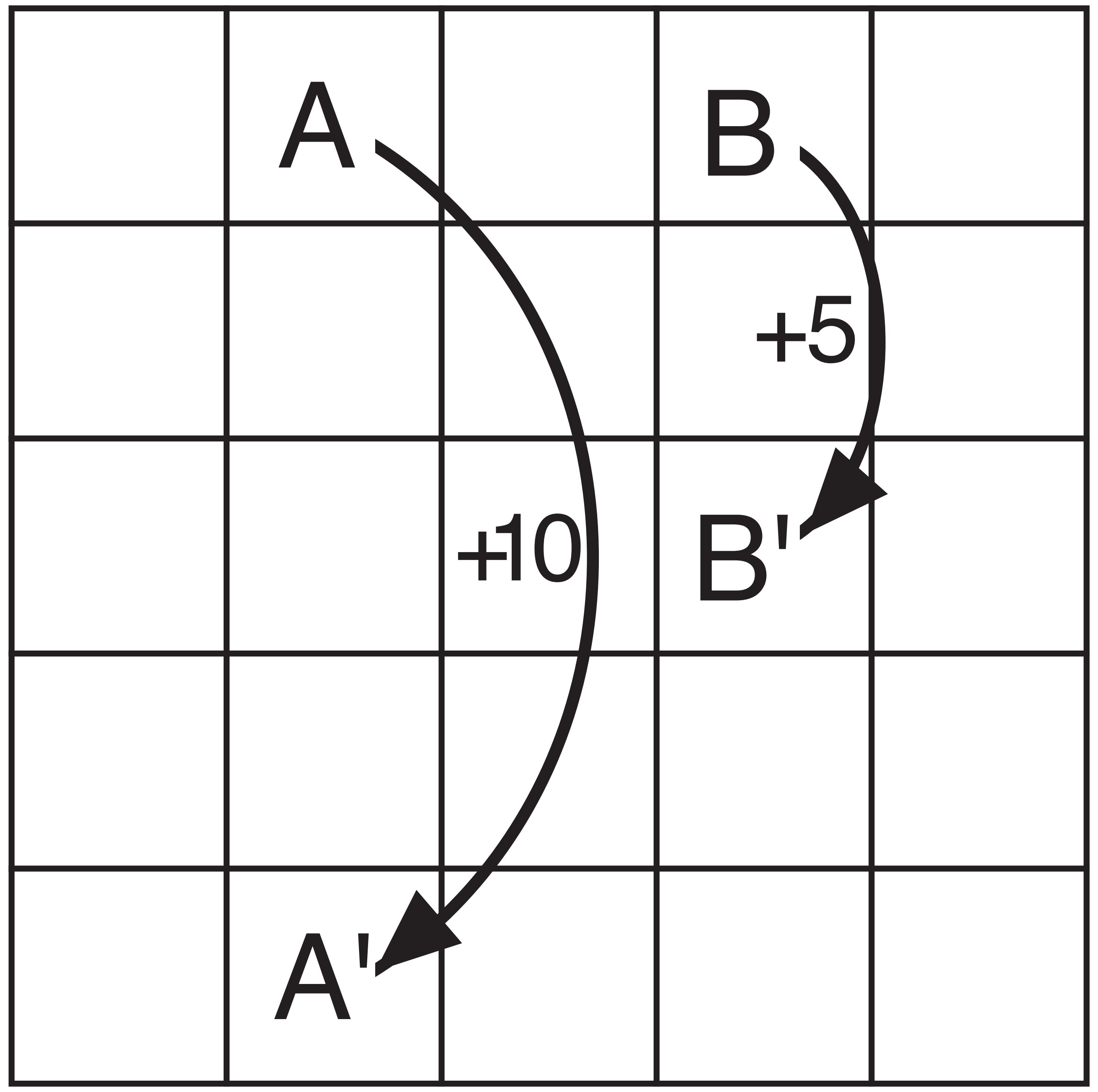
Reward dynamics

-0.5	10	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0

V at $k = 0$

Iterative Policy Evaluation

$$V(s_{3,1}) = \pi(\mathsf{n})[-1+\gamma V(s_{3,1})] + \pi(\mathsf{w})[-1+\gamma \mathbf{V(s_{2,1})}] + \pi(\mathsf{s})[0+\gamma V(s_{3,2})] + \pi(\mathsf{e})[0+\gamma V(s_{4,1})]$$
$$= 0.25[-1+0.9(0)] + 0.25[0+0.9(\mathbf{10})] + 0.25[0+0.9(0)] + 0.25[0+0.9(0)]$$

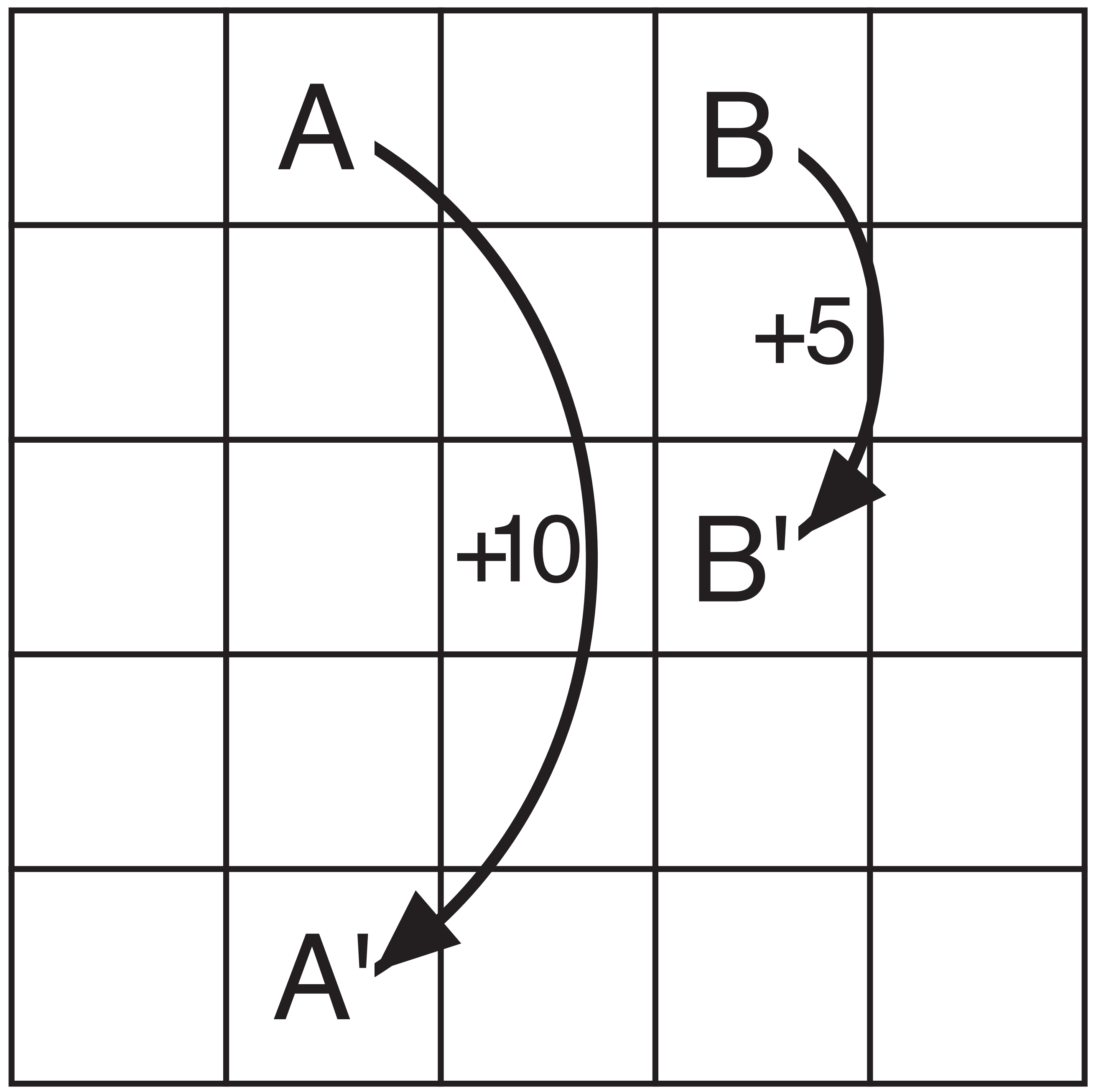


Reward dynamics

-0.5	10	2	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0

V at $k = 0$

Iterative Policy Evaluation in GridWorld

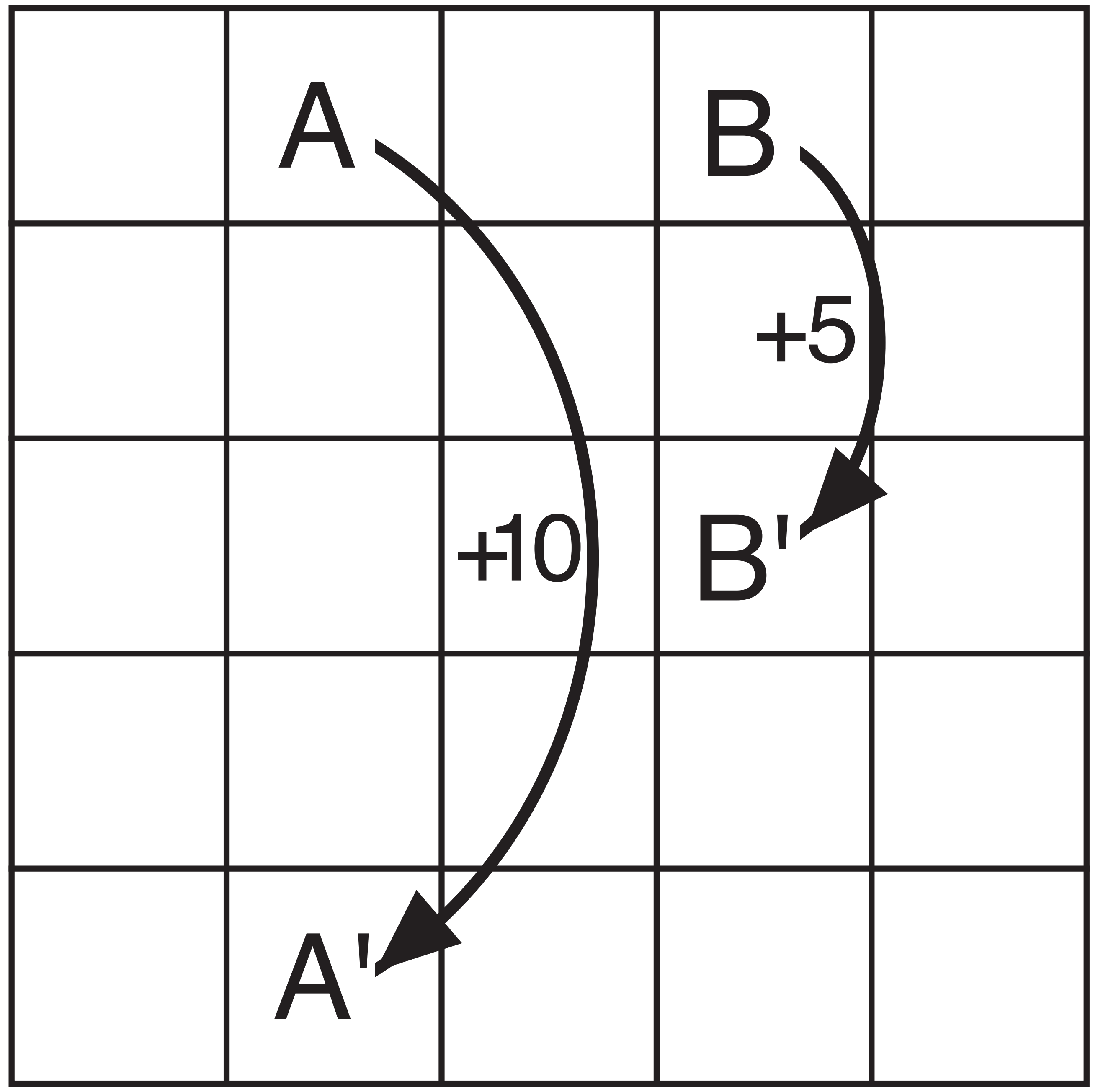


Reward dynamics

-0.5	10	2	5	0.6
-0.3	2.1	0.9	1.3	0.2
-0.3	0.4	0.3	0.4	-0.1
-0.3	0.0	0.0	0.1	-0.2
-0.5	-0.3	-0.3	-0.3	-0.6

V at $k = 1$

Iterative Policy Evaluation in GridWorld

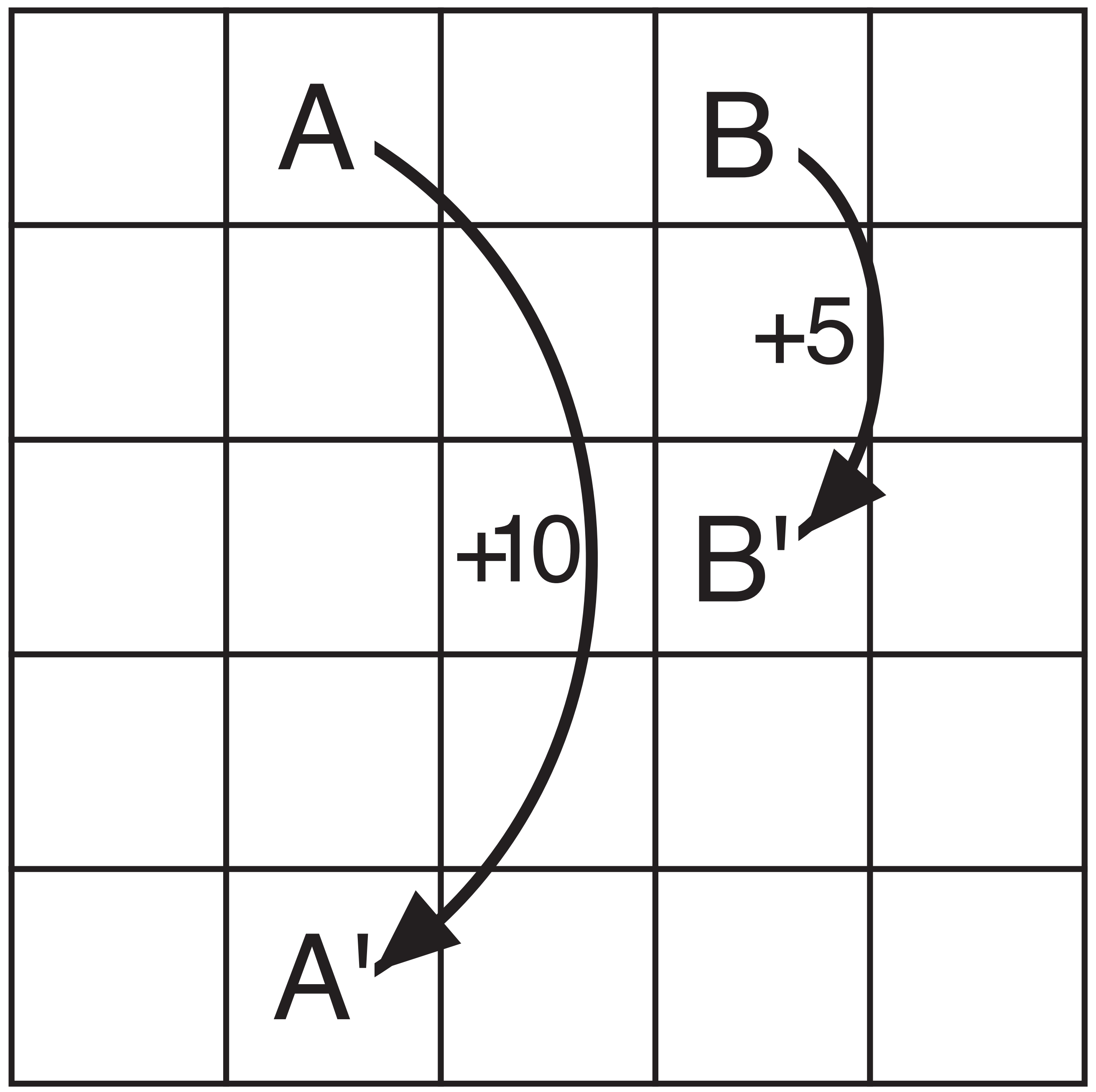


Reward dynamics

1.4	9.7	3.7	5.3	1.0
0.4	2.5	1.8	1.7	0.4
-0.2	0.6	0.6	0.5	-0.1
-0.5	0.0	0.0	0.0	-0.5
-1.0	-0.6	-0.5	-0.5	-1.0

V at $k = 2$

Iterative Policy Evaluation in GridWorld



Reward dynamics

3.4	8.9	4.5	5.3	1.5
1.6	3.0	2.3	1.9	0.6
0.1	0.8	0.7	0.4	-0.4
-1.0	-0.4	-0.3	-0.6	-1.2
-1.9	-1.3	-1.2	-1.4	-2.0

V at $k = 10\,000$

Optimality

- **Question:** What is an **optimal** policy?
- A policy π is (weakly) **better** than a policy π' if it is better **for all** $s \in \mathcal{S}$:

$$\pi \geq \pi' \iff v_\pi(s) \geq v_{\pi'}(s) \quad \forall s \in \mathcal{S}.$$

- An **optimal** policy π_* is weakly better than **every other policy**
  - **Question:** Is an optimal policy guaranteed to exist for a given MDP?
- All optimal policies share the **same state-value function**: (**why?**)

$$v_*(s) \doteq \max_\pi v_\pi(s)$$

- Also the same **action-value function**:

$$q_*(s, a) \doteq \max_\pi q_\pi(s, a)$$

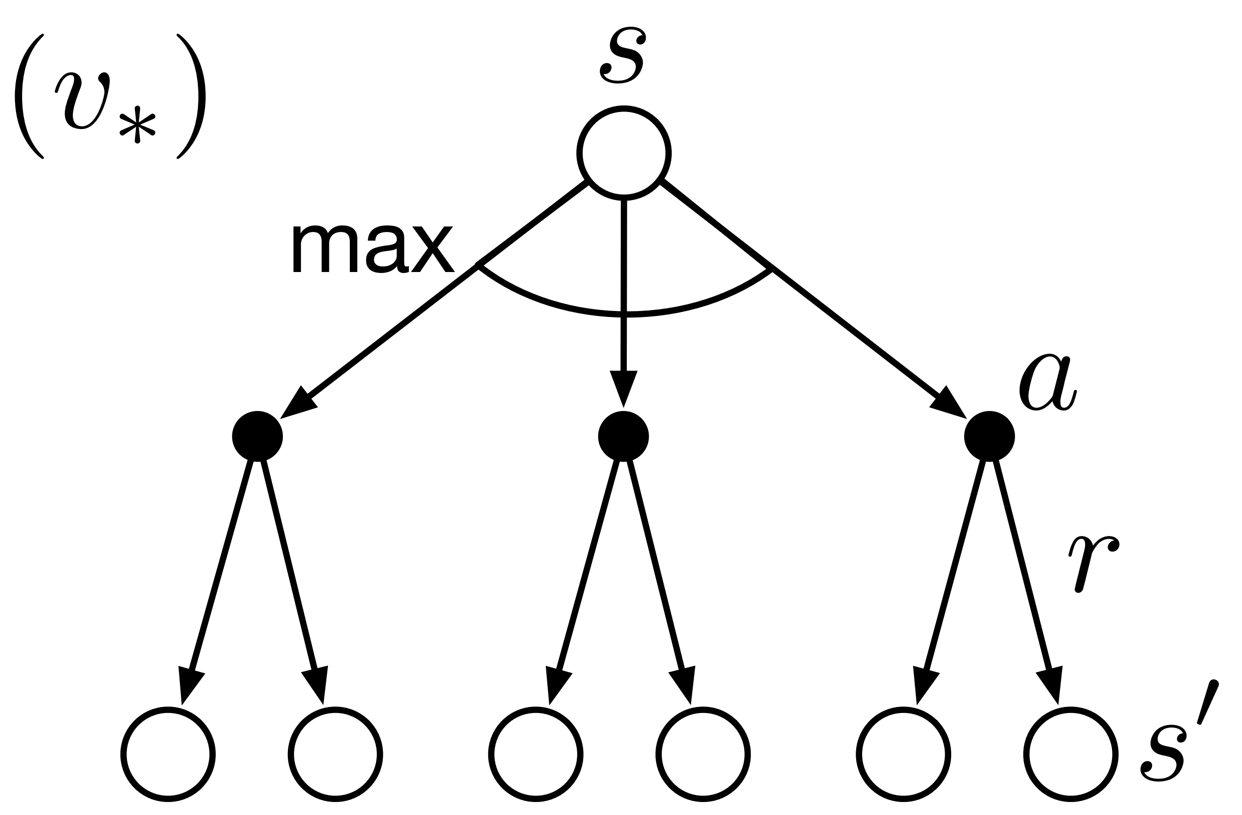
Bellman Optimality Equations

- v_* must satisfy the Bellman equation too
- In fact, it can be written in a special, **policy-free** way because we know that every state value is **maximized** by π_*:

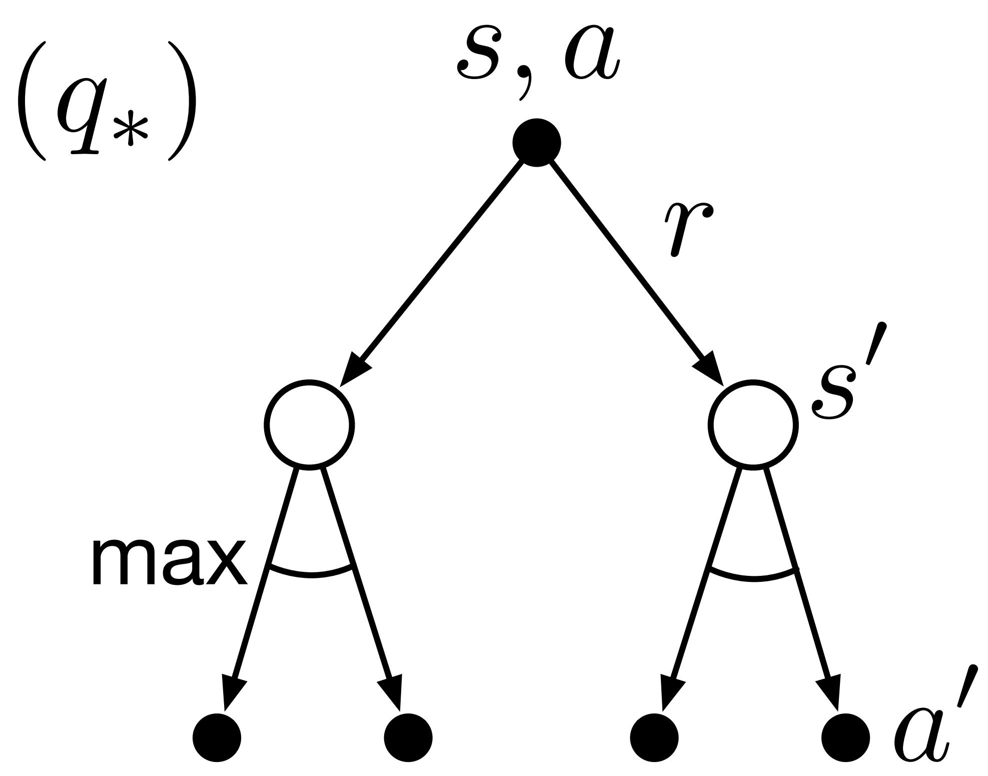
$$
\begin{aligned}
v_*(s) &= \max_a q_{\pi_*}(s, a) \\
&= \max_a \mathbb{E}_{\pi_*}[G_t \mid S_t = s, A_t = a] \\
&= \max_a \mathbb{E}_{\pi_*}[R_{t+1} + \gamma G_{t+1} \mid S_t = s, A_t = a] \\
&= \max_a \mathbb{E}[R_{t+1} + \gamma v_*(S_{t+1}) \mid S_t = s, A_t = a] \\
&= \max_a \sum_{s',r} p(s', r \mid s, a)[r + \gamma v_*(s')]
\end{aligned}
$$

Bellman Optimality Equations

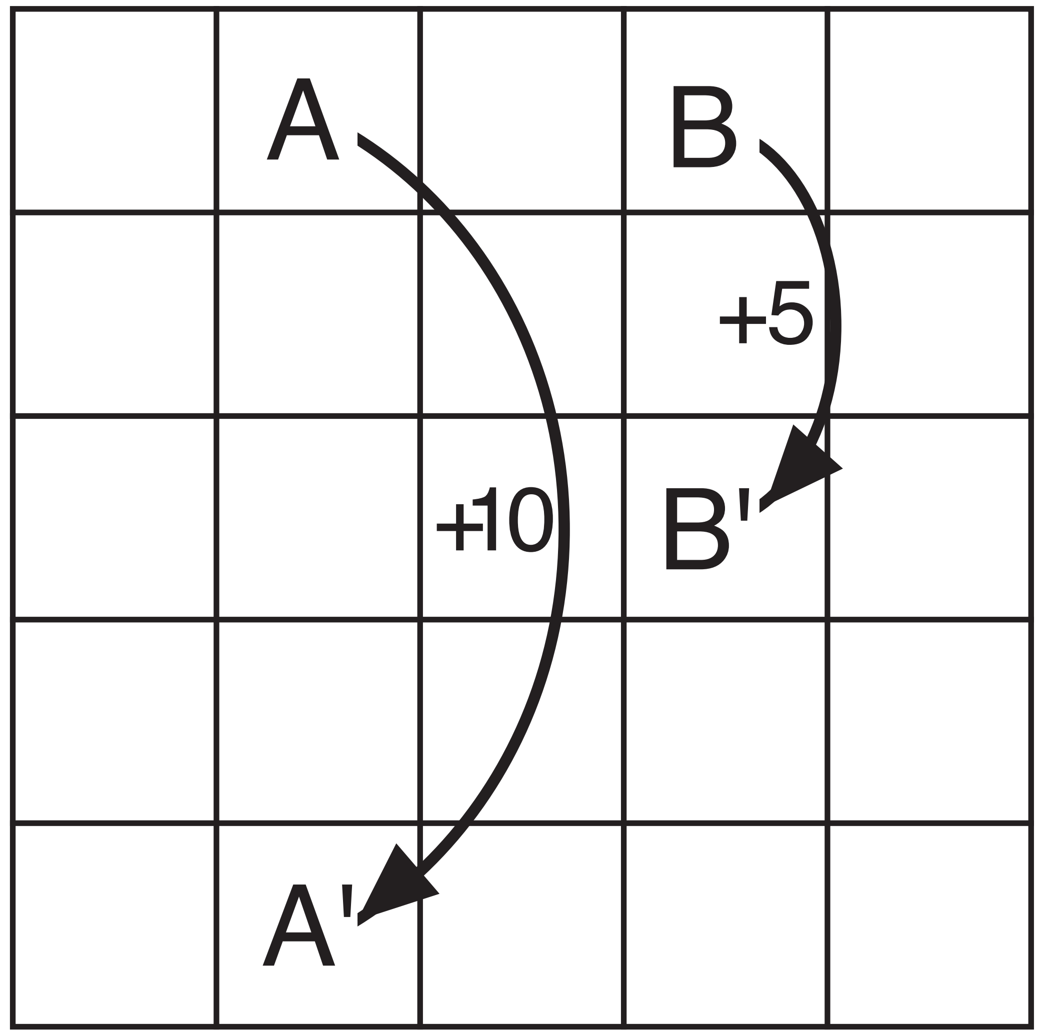
$$\begin{aligned} v_*(s) &= \max_a \mathbb{E}[R_{t+1} + \gamma v_*(S_{t+1}) \,|\, S_t = s, A_t = a] \\ &= \max_a \sum_{s',r} p(s', r \,|\, s, a)[r + \gamma v_*(s')] \end{aligned}$$



$$\begin{aligned} q_*(s,a) &= \mathbb{E}\left[R_{t+1} + \gamma \max_{a'} q_*(S_{t+1}, a') \,\Big|\, S_t = s, A_t = a\right] \\ &= \sum_{s',r} p(s', r \,|\, s, a)\left[r + \gamma \max_{a'} q_*(s', a')\right] \end{aligned}$$



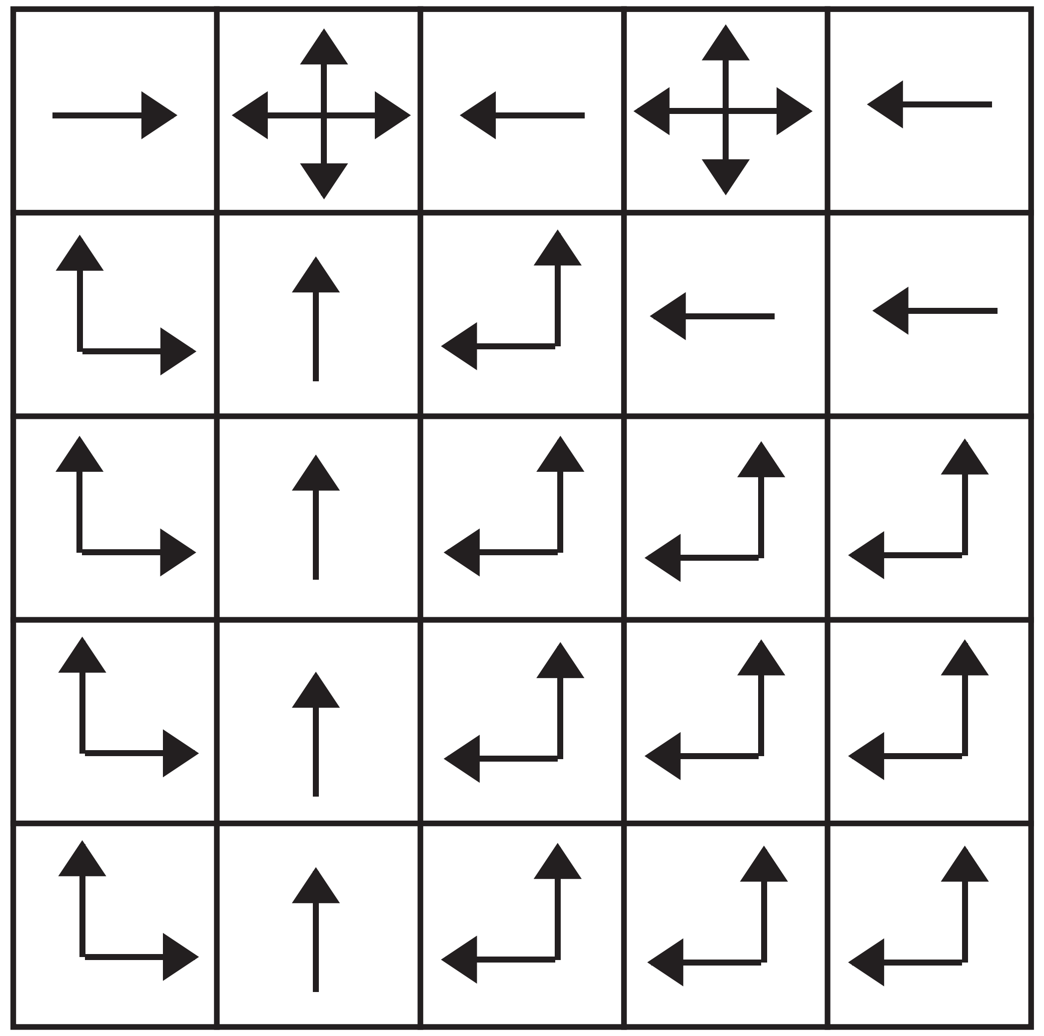
Optimal GridWorld



Gridworld

22.0	24.4	22.0	19.4	17.5
19.8	22.0	19.8	17.8	16.0
17.8	19.8	17.8	16.0	14.4
16.0	17.8	16.0	14.4	13.0
14.4	16.0	14.4	13.0	11.7

v_*



π_*

Policy Improvement Theorem

Theorem:

Let π and π' be any pair of deterministic policies.

If $q_\pi(s, \pi'(s)) \geq v_\pi(s) \quad \forall s \in \mathcal{S}$,

then $v_{\pi'}(s) \geq v_\pi(s) \quad \forall s \in \mathcal{S}$.

If you are never worse off **at any state** by following π' for **one step** and then following π forever after, then following π' **forever** has a higher expected value **at every state**.

Policy Improvement Theorem Proof

$$
\begin{aligned}
v_\pi(s) &\le q_\pi(s, \pi'(s)) \\
&= \mathbb{E}[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s, A_t = \pi'(s)] \\
&= \mathbb{E}_{\pi'}[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s] \\
&\le \mathbb{E}_{\pi'}[R_{t+1} + \gamma q_\pi(S_{t+1}, \pi'(S_{t+1})) \mid S_t = s] \\
&= \mathbb{E}_{\pi'}[R_{t+1} + \gamma \mathbb{E}_{\pi'}[R_{t+2} + \gamma v_\pi(S_{t+2}) | S_{t+1}, A_{t+1} = \pi'(S_{t+1})] \mid S_t = s] \\
&= \mathbb{E}_{\pi'}\big[R_{t+1} + \gamma R_{t+2} + \gamma^2 v_\pi(S_{t+2}) \bigm| S_t = s\big] \\
&\le \mathbb{E}_{\pi'}\big[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 v_\pi(S_{t+3}) \bigm| S_t = s\big] \\
&\;\;\vdots \\
&\le \mathbb{E}_{\pi'}\big[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 R_{t+4} + \cdots \bigm| S_t = s\big] \\
&= v_{\pi'}(s).
\end{aligned}
$$

Greedy Policy Improvement

Given any policy π, we can construct a new greedy policy π' that is guaranteed to be **at least as good**:

$$\begin{aligned}
\pi'(s) &\doteq \arg\max_a q_\pi(s, a) \\
&= \arg\max_a \mathbb{E}[R_{t+1} + \gamma v_\pi(S_{T+1}) \mid S_t = s, A_t = a] \\
&= \arg\max_a \sum_{s', r} p(s', r \mid s, a) \big[r + \gamma v_\pi(s')\big] .
\end{aligned}$$

- If this new policy is **not better** than the old policy, then $v_\pi(s) = v_{\pi'}(s)$ for all $s \in \mathcal{S}$ (**why?**)
- Also means that the new (and old) policies are **optimal** (**why?**)

Policy Iteration

$$\pi_0 \xrightarrow{\text{E}} v_{\pi_0} \xrightarrow{\text{I}} \pi_1 \xrightarrow{\text{E}} v_{\pi_1} \xrightarrow{\text{I}} \pi_2 \xrightarrow{\text{E}} \cdots \xrightarrow{\text{I}} \pi_* \xrightarrow{\text{E}} v_*$$

Policy Iteration (using iterative policy evaluation) for estimating $\pi \approx \pi_*$

1. Initialization
   $V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

2. Policy Evaluation
   Loop:
   - $\Delta \leftarrow 0$
   - Loop for each $s \in \mathcal{S}$:
     - $v \leftarrow V(s)$
     - $V(s) \leftarrow \sum_{s',r} p(s', r \mid s, \pi(s))\big[r + \gamma V(s')\big]$
     - $\Delta \leftarrow \max(\Delta, |v - V(s)|)$

   until $\Delta < \theta$ (a small positive number determining the accuracy of estimation)

3. Policy Improvement
   policy-stable $\leftarrow$ *true*
   For each $s \in \mathcal{S}$:
   - *old-action* $\leftarrow \pi(s)$
   - $\pi(s) \leftarrow \arg\max_a \sum_{s',r} p(s', r \mid s, a)\big[r + \gamma V(s')\big]$
   - If *old-action* $\neq \pi(s)$, then *policy-stable* $\leftarrow$ *false*

   If *policy-stable*, then stop and return $V \approx v_*$ and $\pi \approx \pi_*$; else go to 2

This is a lot of iterations!
Is it necessary to run to completion?

Value Iteration

Value iteration **interleaves** the estimation and improvement steps:

$$v_{k+1}(s) \doteq \max_a \mathbb{E}\left[R_{t+1} + \gamma v_k(S_{t+1}) \,|\, S_t = s, A_t = a\right]$$
$$= \max_a \sum_{s',r} p(s', r \,|\, s, a)\left[r + \gamma v_k(s')\right]$$

Value Iteration, for estimating $\pi \approx \pi_*$

Algorithm parameter: a small threshold $\theta > 0$ determining accuracy of estimation
Initialize $V(s)$, for all $s \in \mathcal{S}^+$, arbitrarily except that $V(terminal) = 0$

Loop:
| $\Delta \leftarrow 0$
| Loop for each $s \in \mathcal{S}$:
| $v \leftarrow V(s)$
| $V(s) \leftarrow \max_a \sum_{s',r} p(s', r \,|\, s, a)\left[r + \gamma V(s')\right]$
| $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$

Output a deterministic policy, $\pi \approx \pi_*$, such that
$\pi(s) = \arg\max_a \sum_{s',r} p(s', r \,|\, s, a)\left[r + \gamma V(s')\right]$

Summary

- An **optimal policy** has higher state value than any other policy **at every state**
- A policy's state-value function can be computed by **iterating** an **expected update** based on the Bellman equation
- Given any policy π, we can compute a **greedy improvement** π' by choosing highest expected value action based on v_π
- **Policy iteration:** Repeat:

  Greedy improvement using v_π, then recompute v_π
- **Value iteration:** Repeat:

  Recompute v_π by assuming greedy improvement at every update