Bayesian Games

CMPUT 654: Modelling Human Strategic Behaviour

S&LB §6.3

Lecture Outline

1. Recap
2. Bayesian Game Definitions
3. Strategies and Expected Utility
4. Bayes-Nash Equilibrium

Recap: Repeated Games

- A **repeated game** is one in which agents play the same normal form game (the **stage game**) multiple times
- **Finitely repeated:** Can represent as an **imperfect information extensive form** game
- **Infinitely repeated:** Life gets more complicated
  - Payoff to the game: either **average** or **discounted** reward
  - **Pure strategies** map from **entire previous history** to action
- Need to define the **expected utility** of pure strategies xsas well as **pure strategies** before we can leverage our existing definitions

Fun Game!

- Everyone should have a slip of paper with 2 **dollar values** on it
- Play a sealed-bid first-price auction with three other people
  - If you win, utility is your first dollar value minus your bid
  - If you lose, utility is 0
- Play again with the same neighbours, same valuation
- Then play again with same neighbours, valuation #2
- **Question:** How can we model this interaction as a game?

Payoff Uncertainty

- Up until now, we have assumed that the following are always **common knowledge**:
  - **Number** of players
  - **Pure strategies** available to each player
  - **Payoffs** associated with each pure strategy profile
- Bayesian games are games in which there is uncertainty about the very **game being played**

Bayesian Games

We will assume the following:

1. In every possible game, number of **actions** available to each player is the same; they differ only in their **payoffs**

2. Every agent's **beliefs** are posterior beliefs obtained by conditioning a **common prior** distribution on private signals.

There are at least three ways to define a Bayesian game.

Bayesian Games via Information Sets

Definition:
A **Bayesian game** is a tuple (N,G,P,I), where

- N is a set of ***agents***
- G is a set of ***games*** with N agents such that if $g,g' \in G$ then for each agent $i \in N$ the pure strategies available to i in g are identical to the pure strategies available to i in g'
- $P \in \mathbf{\Delta}(G)$ is a **common prior** over games in G
- $I=(I_1, I_2, \ldots, I_n)$ is a tuple of **partitions** over G, one for each agent

Information Sets Example

$I_{2,1}$ $I_{2,2}$

$I_{1,1}$

MP

$2, 0$	$0, 2$
$0, 2$	$2, 0$

$p = 0.3$

PD

$2, 2$	$0, 3$
$3, 0$	$1, 1$

$p = 0.1$

$I_{1,2}$

Coord

$2, 2$	$0, 0$
$0, 0$	$1, 1$

$p = 0.2$

BoS

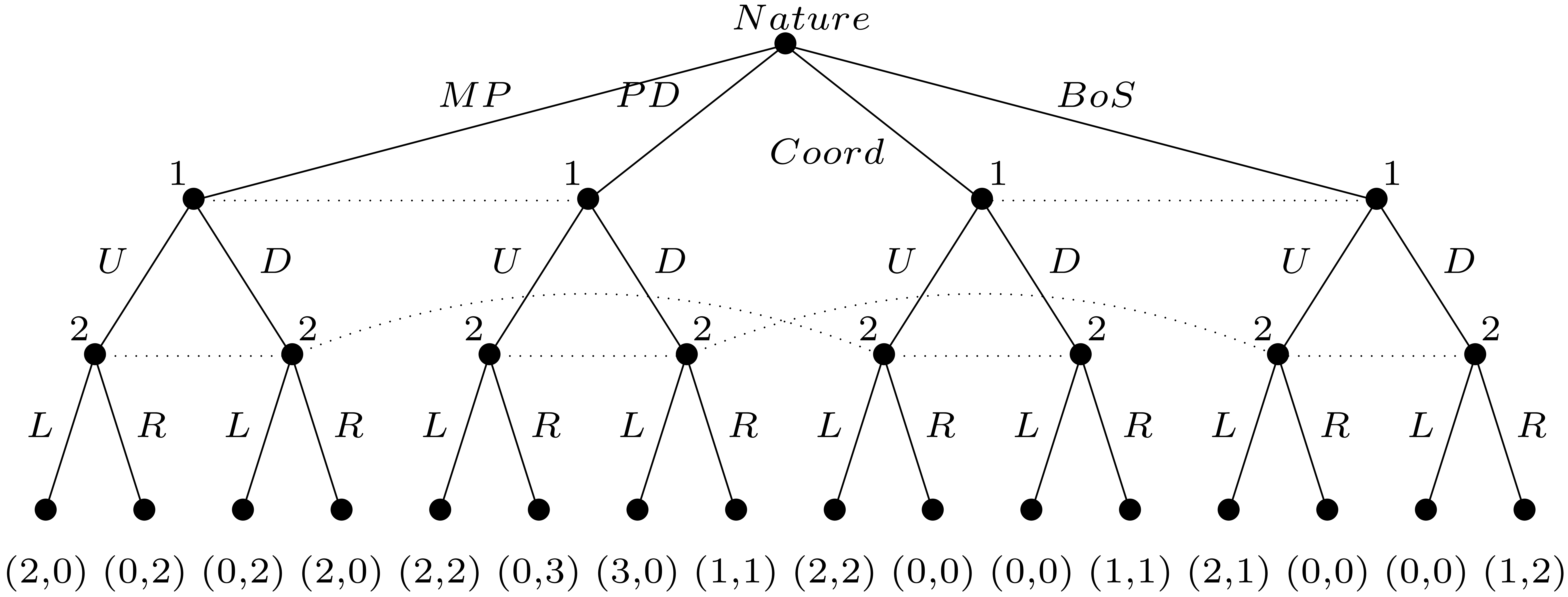
$2, 1$	$0, 0$
$0, 0$	$1, 2$

$p = 0.4$

Bayesian Games via Imperfect Information with Nature

- Could instead have a special agent **Nature** plays according to a commonly-known mixed strategy
- **Nature** chooses the game at the outset
- Cumbersome for simultaneous-move Bayesian games
- Makes more sense for sequential-move Bayesian games, especially when players learn from other players' moves

Imperfect Information with Nature Example



Bayesian Games via Epistemic Types

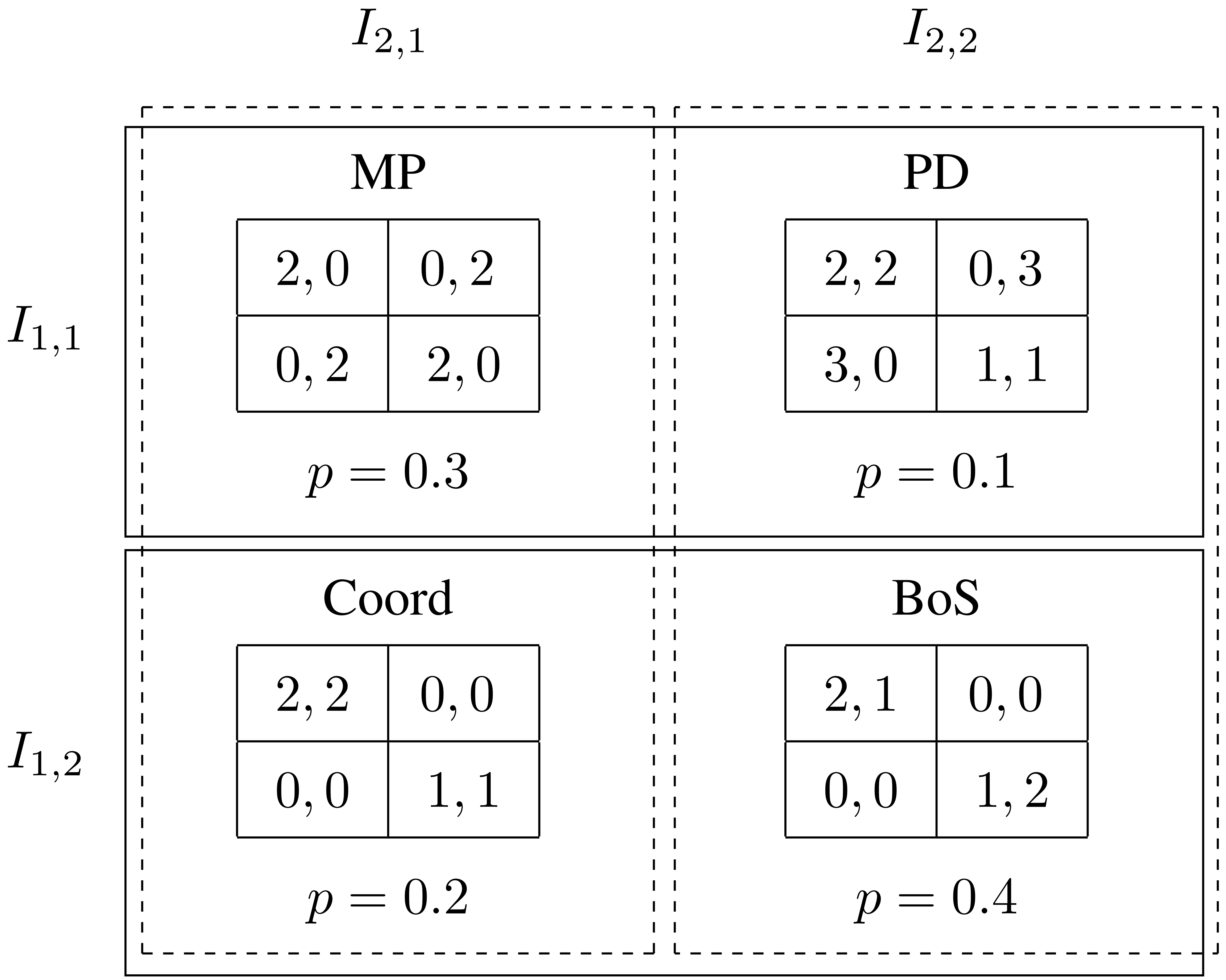
Definition:
A Bayesian game is a tuple (N, A, Θ, p, u) where

- N is a set of n ***players***
- $A = A_1 \times A_2 \times \ldots \times A_n$ is the set of **action profiles**
    - A_i is the **action set** for player i
- $\Theta = \Theta_1 \times \Theta_2 \times \ldots \times \Theta_n$ is the set of **type profiles**
    - Θ_1 is the **type space** of player i
- p is a **prior distribution** over type profiles
- $u = (u_1, u_2, \ldots, u_n)$ is a tuple of **utility functions**, one for each player
    - $u_i : A \times \Theta \to \mathbb{R}$

What is a Type?

- All of the elements in the previous definition are **common knowledge**
    - Parameterizes utility functions in a **known way**
- Every player knows their **own type**
- Type encapsulates all of the knowledge that a player has that is **not common knowledge**:
    - Beliefs about **own payoffs**
    - But also beliefs about **other player's payoffs**
    - But *also* beliefs about **other player's beliefs** about own payoffs

Epistemic Types Example



a_1	a_2	θ_1	θ_2	u_1	u_2
U	L	$\theta_{1,1}$	$\theta_{2,1}$	2	0
U	L	$\theta_{1,1}$	$\theta_{2,2}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,1}$	2	2
U	L	$\theta_{1,2}$	$\theta_{2,2}$	2	1
U	R	$\theta_{1,1}$	$\theta_{2,1}$	0	2
U	R	$\theta_{1,1}$	$\theta_{2,2}$	0	3
U	R	$\theta_{1,2}$	$\theta_{2,1}$	0	0
U	R	$\theta_{1,2}$	$\theta_{2,2}$	0	0

a_1	a_2	θ_1	θ_2	u_1	u_2
D	L	$\theta_{1,1}$	$\theta_{2,1}$	0	2
D	L	$\theta_{1,1}$	$\theta_{2,2}$	3	0
D	L	$\theta_{1,2}$	$\theta_{2,1}$	0	0
D	L	$\theta_{1,2}$	$\theta_{2,2}$	0	0
D	R	$\theta_{1,1}$	$\theta_{2,1}$	2	0
D	R	$\theta_{1,1}$	$\theta_{2,2}$	1	1
D	R	$\theta_{1,2}$	$\theta_{2,1}$	1	1
D	R	$\theta_{1,2}$	$\theta_{2,2}$	1	2

Strategies

- **Pure strategy**: mapping from agent's type to an action

$$s_i : \Theta_i \to A_i$$

- **Mixed strategy**: distribution over an agent's pure strategies

$$s_i \in \Delta(A^{\Theta_i})$$

  - *or:* mapping from type to distribution over actions

$$s_i : \Theta_i \to \Delta(A)$$

  - **Question:** is this equivalent? Why or why not?

  - We can use conditioning notation for the probability that *i* plays a_i given that their type is θ_i

$$s_i(a_i \mid \theta_i)$$

Expected Utility

The agent's expected utility is different depending on **when** they compute it, because it is taken with respect to different **distributions**.

Three relevant timeframes:

1. ***Ex-ante***: agent knows **nobody's** type
2. ***Ex-interim***: agent knows **own** type but not others'
3. ***Ex-post***: agent knows **everybody's** type

Ex-post Expected Utility

Definition:
Agent i's ***ex-post* expected utility** in a Bayesian game (N, A, Θ, p, u), where the agents' strategy profile is s and the agents' type profile is θ, is defined as

$$EU_i(s, \theta) = \sum_{a \in A} \left(\prod_{j \in N} s_j(a_j \mid \theta_j) \right) u_i(a) \, .$$

- The only source of uncertainty is in which **actions** will be realized from the mixed strategies.

Ex-interim Expected Utility

Definition:
Agent *i*'s ***ex-interim* expected utility** in a Bayesian game (N, A, Θ, p, u), where the agents' strategy profile is *s* and *i*'s type is θ_i, is defined as

$$EU_i(s, \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i} \mid \theta_i) \sum_{a \in A} \left(\prod_{j \in N} s_j(a_j \mid \theta_j) \right) u_i(a),$$

or equivalently as

$$EU_i(s, \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} p(\theta_{-i} \mid \theta_i) EU_i(s, (\theta_i, \theta_{-i})).$$

- Uncertainty over both the **actions** realized from the mixed strategy profile, and the **types** of the other agents.

Ex-ante Expected Utility

Definition:
Agent *i*'s ***ex-ante* expected utility** in a Bayesian game (N, A, Θ, p, u), where the agents' strategy profile is *s*, is defined as

$$EU_i(s) = \sum_{\theta \in \Theta} p(\theta) \sum_{a \in A} \left(\prod_{j \in N} s_j(a_j \mid \theta_j) \right) u_i(a),$$

or equivalently as

$$EU_i(s) = \sum_{\theta_i \in \Theta_i} p(\theta_i) EU_i(s, \theta_i)\,.$$

or again equivalently as

$$EU_i(s) = \sum_{\theta \in \Theta} p(\theta) EU_i(s, \theta)\,.$$

Question:

Why are these three expressions equivalent?

Best Response

Question: What is a **best response** in a Bayesian game?

Definition:
The set of agent *i*'s **best responses** to mixed strategy profile s_{-i} are given by

$$BR_i(s_{-i}) = \arg\max_{s_i' \in S_i} EU_i(s_i', s_{-i})\,.$$

Question: Why is this defined using *ex-ante* expected utility?

Bayes-Nash Equilibrium

Question: What is the **induced normal form** for a Bayesian game?

Question: What is a **Nash equilibrium** in a Bayesian game?

Definition:
A **Bayes-Nash equilibrium** is a mixed strategy profile s that satisfies

$$\forall i \in N : s_i \in BR_i(s_{-i}) \, .$$

Ex-post Equilibrium

Definition:
An ***ex-post* equilibrium** is a mixed strategy profile *s* that satisfies

$$\forall \theta \in \Theta \ \ \forall i \in N : s_i \in \arg\max_{s_i'} EU((s_i', s_{-i}), \theta) \,.$$

- Ex-post equilibrium is similar to dominant-strategy equilibrium, but neither implies the other
  - **Dominant strategy equilibrium**: agents need not have accurate beliefs about others' **strategies**
  - ***Ex-post* equilibrium:** agents need not have accurate beliefs about others' **types**

Dominant Strategy Equilibrium vs Ex-post Equilibrium

- **Question:** What is a **dominant strategy** in a Bayesian game?

- Example game in which a dominant strategy equilibrium is not an ex-post equilibrium:

$$
\begin{aligned}
N &= \{1,2\} & & \\
A_i &= \Theta_i = \{H, L\} & & \forall i \in N \\
p(\theta) &= 0.5 & & \forall \theta \in \Theta \\
u_i(a, \theta) &= \begin{cases} 10 \text{ if } a_i = \theta_{-i} = \theta_i, \\ 2 \quad \text{if } a_i = \theta_{-i} \neq \theta_i, \\ 0 \quad \text{otherwise.} \end{cases} & & \forall i \in N
\end{aligned}
$$

Summary

- Bayesian games represent settings in which there is uncertainty about the **very game being played**
- Can be defined as **game of imperfect information** with a **Nature** player, *or* as a **partition and prior** over games
- Can be defined using **epistemic types**
- **Expected utility** evaluates against three different distributions:
    - *ex-ante*, *ex-interim*, and *ex-post*
- **Bayes-Nash equilibrium** is the usual solution concept
    - ***Ex-post* equilibrium** is a stronger solution concept