# Monte Carlo Control

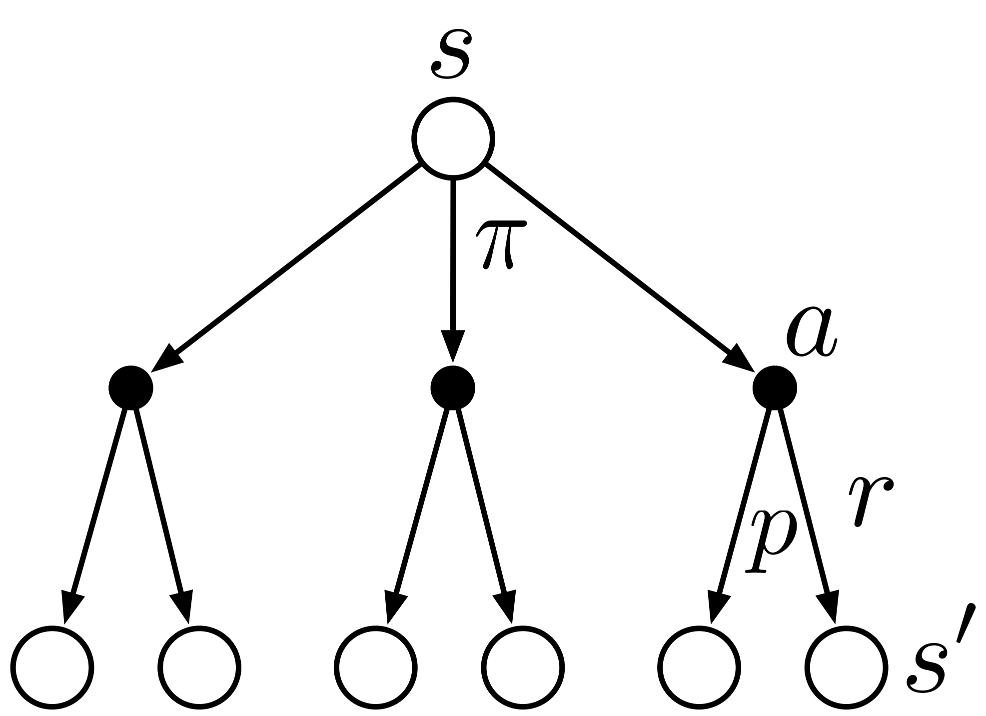
CMPUT 366: Intelligent Systems

S&B §5.3-5.5, 5.7

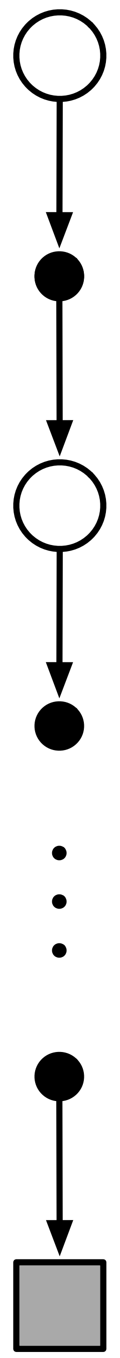
# Lecture Outline

1. Recap
2. Estimating Action Values
3. Monte Carlo Control
4. Importance Sampling
5. Off-Policy Monte Carlo Control

# Recap: Monte Carlo vs. Dynamic Programming




- **Iterative policy evaluation** uses the estimates of the **next state's** value to update the value of this state
    - Only needs to compute a **single transition** to update a state's estimate
- **Monte Carlo** estimate of each state's value is **independent** from estimates of **other states'** values
    - Needs the **entire episode** to compute an update
    - Can focus on evaluating a **subset of states** if desired



# First-visit Monte Carlo Prediction

**First-visit MC prediction, for estimating $V \approx v_\pi$**

Input: a policy $\pi$ to be evaluated

Initialize:

$V(s) \in \mathbb{R}$, arbitrarily, for all $s \in \mathcal{S}$

$Returns(s) \leftarrow$ an empty list, for all $s \in \mathcal{S}$

Loop forever (for each episode):

Generate an episode following $\pi$: $S_0, A_0, R_1, S_1, A_1, R_2, \ldots, S_{T-1}, A_{T-1}, R_T$

$G \leftarrow 0$

Loop for each step of episode, $t = T-1, T-2, \ldots, 0$:

$G \leftarrow \gamma G + R_{t+1}$

Unless $S_t$ appears in $S_0, S_1, \ldots, S_{t-1}$:

Append $G$ to $Returns(S_t)$

$V(S_t) \leftarrow \text{average}(Returns(S_t))$

# Control vs. Prediction

- **Prediction:** estimate the value of states and/or actions given some **fixed policy** $\pi$
- **Control:** estimate an **optimal policy**

# Estimating Action Values

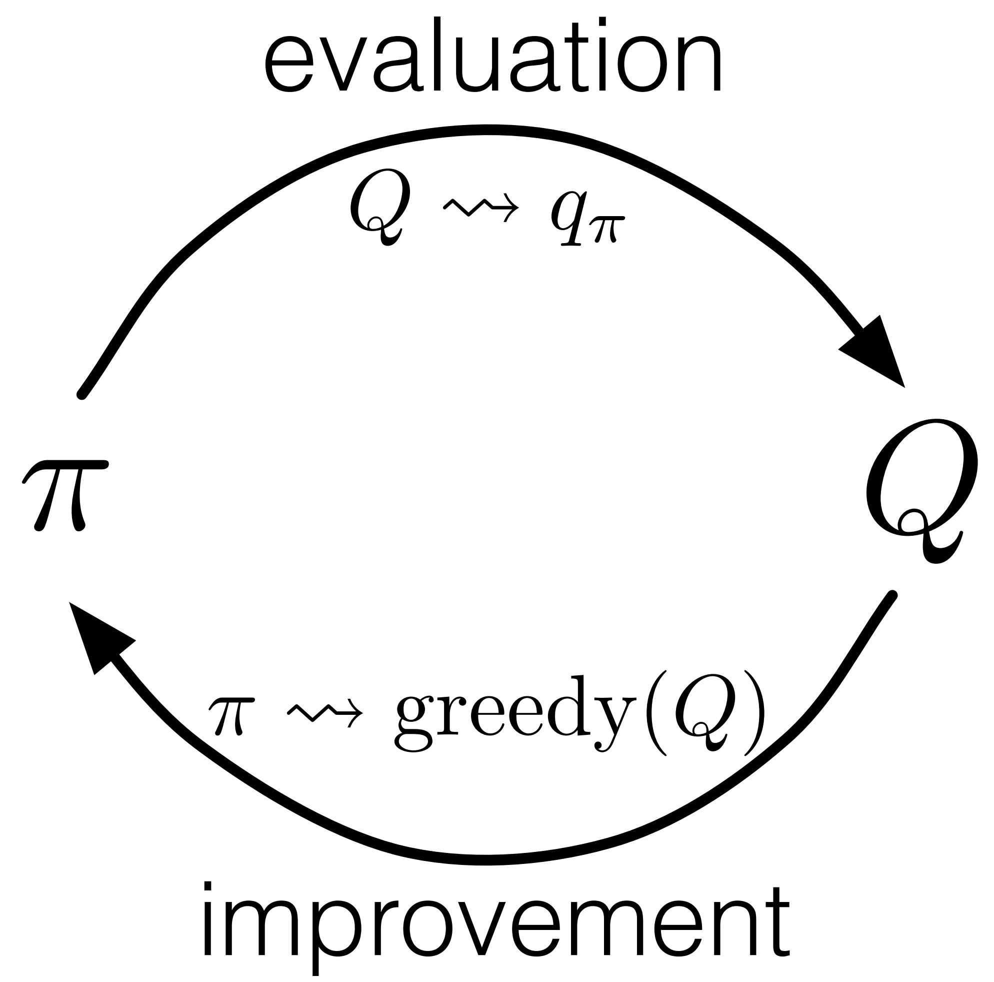
- When we know the **dynamics** $p(s', r \mid s, a)$, an estimate of **state values** is sufficient to determine a good **policy**:
    - Choose the action that gives the best combination of reward and next-state value
- If we don't know the dynamics, state values are **not enough**
    - To estimate a good policy, we need an **explicit** estimate of **action values**

# Exploring Starts

- We can just run first-visit Monte Carlo and approximate the returns to each **state-action pair**
- **Question:** What do we do about state-action pairs that are **never visited**?
  - If the current policy $\pi$ never selects an action $a$ from a state $s$, then Monte Carlo can't estimate its value
- **Exploring starts assumption:**
  - Every episode **starts** at a state-action pair $S_0, A_0$
  - **Every pair** has a positive probability of being selected for a start

# Monte Carlo Control

Monte Carlo control can be used for **policy iteration**:




$$\pi_0 \xrightarrow{\mathrm{E}} q_{\pi_0} \xrightarrow{\mathrm{I}} \pi_1 \xrightarrow{\mathrm{E}} q_{\pi_1} \xrightarrow{\mathrm{I}} \pi_2 \xrightarrow{\mathrm{E}} \cdots \xrightarrow{\mathrm{I}} \pi_* \xrightarrow{\mathrm{E}} q_*$$

# Monte Carlo Control with Exploring Starts

**Monte Carlo ES (Exploring Starts), for estimating $\pi \approx \pi_*$**

Initialize:
- $\pi(s) \in \mathcal{A}(s)$ (arbitrarily), for all $s \in \mathcal{S}$
- $Q(s,a) \in \mathbb{R}$ (arbitrarily), for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$
- $Returns(s,a) \leftarrow$ empty list, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$

Loop forever (for each episode):
- Choose $S_0 \in \mathcal{S}$, $A_0 \in \mathcal{A}(S_0)$ randomly such that all pairs have probability $> 0$
- Generate an episode from $S_0, A_0$, following $\pi$: $S_0, A_0, R_1, \ldots, S_{T-1}, A_{T-1}, R_T$
- $G \leftarrow 0$
- Loop for each step of episode, $t = T-1, T-2, \ldots, 0$:
  - $G \leftarrow \gamma G + R_{t+1}$
  - Unless the pair $S_t, A_t$ appears in $S_0, A_0, S_1, A_1 \ldots, S_{t-1}, A_{t-1}$:
    - Append $G$ to $Returns(S_t, A_t)$
    - $Q(S_t, A_t) \leftarrow \text{average}(Returns(S_t, A_t))$
    - $\pi(S_t) \leftarrow \arg\max_a Q(S_t, a)$

**Question:** What unlikely assumptions does this rely upon?

# $\varepsilon$-Soft Policies

- The **exploring starts** assumption ensures that we see **every** state-action pair with positive probability
  - Even if $\pi$ **never** chooses $a$ from state $s$
- Another approach: Simply **force** $\pi$ to (sometimes) choose $a$!
- An **$\epsilon$-soft policy** is one for which $\pi(a \mid s) \geq \epsilon \quad \forall s, a$
- **Example: $\epsilon$-greedy policy**

$$\pi(a \mid s) = \begin{cases} \frac{\epsilon}{|\mathcal{A}|} & \text{if } a \notin \arg\max_a Q(s,a), \\ 1 - \epsilon + \frac{\epsilon}{|\mathcal{A}|} & \text{otherwise.} \end{cases}$$

# Monte Carlo Control w/out Exploring Starts

**On-policy first-visit MC control (for $\varepsilon$-soft policies), estimates $\pi \approx \pi_*$**

Algorithm parameter: small $\varepsilon > 0$

Initialize:

  $\pi \leftarrow$ an arbitrary $\varepsilon$-soft policy

  $Q(s,a) \in \mathbb{R}$ (arbitrarily), for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$

  $Returns(s,a) \leftarrow$ empty list, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$

Repeat forever (for each episode):

  Generate an episode following $\pi$: $S_0, A_0, R_1, \ldots, S_{T-1}, A_{T-1}, R_T$

  $G \leftarrow 0$

  Loop for each step of episode, $t = T-1, T-2, \ldots, 0$:

    $G \leftarrow \gamma G + R_{t+1}$

    Unless the pair $S_t, A_t$ appears in $S_0, A_0, S_1, A_1 \ldots, S_{t-1}, A_{t-1}$:

      Append $G$ to $Returns(S_t, A_t)$

      $Q(S_t, A_t) \leftarrow \text{average}(Returns(S_t, A_t))$

      $A^* \leftarrow \arg\max_a Q(S_t, a)$ (with ties broken arbitrarily)

      For all $a \in \mathcal{A}(S_t)$:

$$\pi(a|S_t) \leftarrow \begin{cases} 1 - \varepsilon + \varepsilon/|\mathcal{A}(S_t)| & \text{if } a = A^* \\ \varepsilon/|\mathcal{A}(S_t)| & \text{if } a \neq A^* \end{cases}$$

# Monte Carlo Control w/out Exploring Starts

**On-policy first-visit MC control (for $\varepsilon$-soft policies), estimates $\pi \approx \pi_*$**

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Repeat forever (for each episode):

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Unless the pair $S_t, A_t$ appears in $S_0, A_0, S_1, A_1 \ldots, S_{t-1}, A_{t-1}$:

Append $G$ to $Returns(S_t, A_t)$

$Q(S_t, A_t) \leftarrow \text{average}(Returns(S_t, A_t))$

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For all $a \in \mathcal{A}(S_t)$:

$$\pi(a|S_t) \leftarrow \begin{cases} 1 - \varepsilon + \varepsilon/|\mathcal{A}(S_t)| & \text{if } a = A^* \\ \varepsilon/|\mathcal{A}(S_t)| & \text{if } a \neq A^* \end{cases}$$

**Question:**

Will this procedure converge to the optimal policy $\pi^*$?

Why or why not?

# Importance Sampling

- **Question:** What was **importance sampling** the last time we studied it (in Supervised Learning?)
- **Monte Carlo sampling:** use samples from the **target** distribution to estimate expectations
- **Importance sampling:** Use samples from **proposal** distribution to estimate expectations of **target** distribution by **reweighting** samples

$$\mathbb{E}[X] = \sum_x f(x)x = \sum_x \frac{g(x)}{g(x)} f(x)x = \sum_x g(x)\frac{f(x)}{g(x)}x \approx \frac{1}{n}\sum_{x_i \sim g} \underbrace{\boxed{\frac{f(x_i)}{g(x_i)}}}_{\text{Importance sampling ratio}} x_i$$

# Off-Policy Prediction via Importance Sampling

**Definition:**
**Off-policy learning** means using data generated by a **behaviour policy** to learn about a distinct **target policy**.

Proposal distribution

Target distribution

# Off-Policy Monte Carlo Prediction

- Generate episodes using **behaviour policy** $b$
- Take **weighted** average of returns to state $s$ over all the episodes containing a visit to $s$ to estimate $v_\pi(s)$
  - Weighed by **importance sampling ratio** of trajectory starting from $S_t = s$ until the end of the episode:

$$\rho_{t:T-1} \doteq \frac{\Pr[A_t, S_{t+1}, \ldots, S_T \,|\, S_t, A_{t:T-1} \sim \pi]}{\Pr[A_t, S_{t+1}, \ldots, S_T \,|\, S_t, A_{t:T-1} \sim b]}$$

# Importance Sampling Ratios for Trajectories

- **Probability of a trajectory** $A_t, S_{t+1}, A_{t+1}, \ldots, S_T$ from $S_t$:

$$\Pr[A_t, S_{t+1}, \ldots, S_T \mid S_t, A_{t:T-1} \sim \pi] =$$

$$\pi(A_t \mid S_t) p(S_{t+1} \mid S_t, A_t) \pi(A_{t+1} \mid S_{t+1}) \ldots p(S_T \mid S_{T-1}, A_{T-1})$$

- **Importance sampling ratio** for a trajectory $A_t, S_{t+1}, A_{t+1}, \ldots, S_T$ from $S_t$:

$$\rho_{t:T-1} \doteq \frac{\prod_{k=t}^{T-1} \pi(A_k \mid S_k) p(S_{k+1} \mid S_k, A_k)}{\prod_{k=t}^{T-1} b(A_k \mid S_k) p(S_{k+1} \mid S_k, A_k)} = \frac{\prod_{k=t}^{T-1} \pi(A_k \mid S_k)}{\prod_{k=t}^{T-1} b(A_k \mid S_k)}$$

# Ordinary vs. Weighted Importance Sampling

- **Ordinary importance sampling:**

$$V(s) \doteq \frac{1}{n} \sum_{i=1}^{n} \rho_{t(s,i):T(i)-1} G_{i,t}$$

- **Weighted importance sampling:**

$$V(s) \doteq \frac{\sum_{i=1}^{n} \rho_{t(s,i):T(i)-1} G_{i,t}}{\sum_{i=1}^{n} \rho_{t(s,i):T(i)-1}}$$

# Example: Ordinary vs. Weighted Importance Sampling for Blackjack

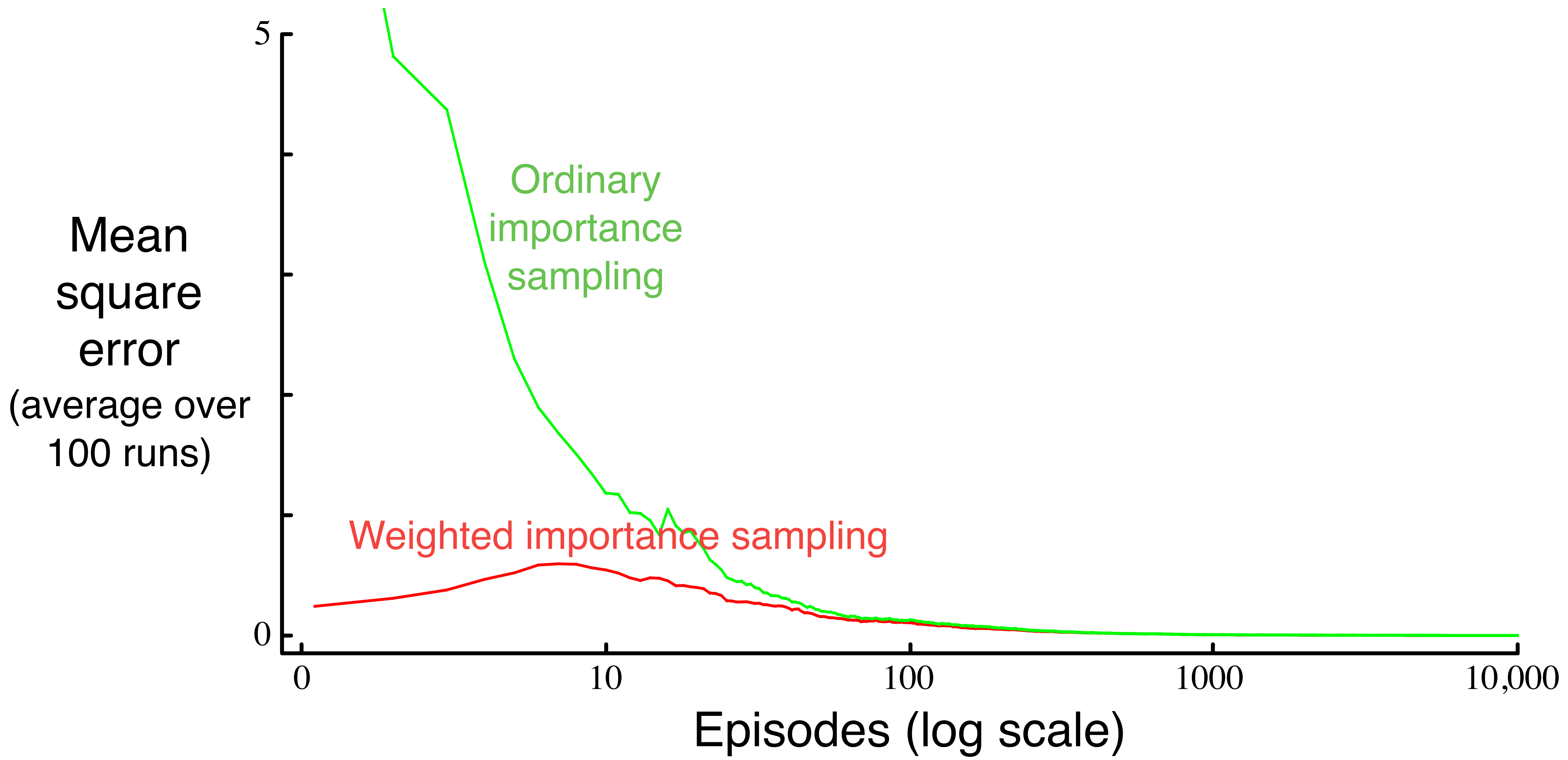


**Figure 5.3:** Weighted importance sampling produces lower error estimates of the value of a single blackjack state from off-policy episodes. ■

(Image: Sutton & Barto, 2018)

# Off-Policy Monte Carlo Prediction

**Off-policy MC prediction (policy evaluation) for estimating $Q \approx q_\pi$**

Input: an arbitrary target policy $\pi$
Initialize, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$:
    $Q(s,a) \in \mathbb{R}$ (arbitrarily)
    $C(s,a) \leftarrow 0$

Loop forever (for each episode):
    $b \leftarrow$ any policy with coverage of $\pi$
    Generate an episode following $b$: $S_0, A_0, R_1, \ldots, S_{T-1}, A_{T-1}, R_T$
    $G \leftarrow 0$
    $W \leftarrow 1$
    Loop for each step of episode, $t = T-1, T-2, \ldots, 0$, while $W \neq 0$:
        $G \leftarrow \gamma G + R_{t+1}$
        $C(S_t, A_t) \leftarrow C(S_t, A_t) + W$
        $Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \frac{W}{C(S_t,A_t)} \left[G - Q(S_t, A_t)\right]$
        $W \leftarrow W \frac{\pi(A_t|S_t)}{b(A_t|S_t)}$

# Off-Policy Monte Carlo Control

**Off-policy MC control, for estimating $\pi \approx \pi_*$**

Initialize, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$:
  $Q(s,a) \in \mathbb{R}$ (arbitrarily)
  $C(s,a) \leftarrow 0$
  $\pi(s) \leftarrow \arg\max_a Q(s,a)$   (with ties broken consistently)

Loop forever (for each episode):
  $b \leftarrow$ any soft policy
  Generate an episode using $b$: $S_0, A_0, R_1, \ldots, S_{T-1}, A_{T-1}, R_T$
  $G \leftarrow 0$
  $W \leftarrow 1$
  Loop for each step of episode, $t = T-1, T-2, \ldots, 0$:
    $G \leftarrow \gamma G + R_{t+1}$
    $C(S_t, A_t) \leftarrow C(S_t, A_t) + W$
    $Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \frac{W}{C(S_t,A_t)} \left[G - Q(S_t, A_t)\right]$
    $\pi(S_t) \leftarrow \arg\max_a Q(S_t, a)$   (with ties broken consistently)
    If $A_t \neq \pi(S_t)$ then exit inner Loop (proceed to next episode)
    $W \leftarrow W \frac{1}{b(A_t|S_t)}$

# Off-Policy Monte Carlo Control

**Off-policy MC control, for estimating $\pi \approx \pi_*$**

Initialize, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$:
    $Q(s,a) \in \mathbb{R}$ (arbitrarily)
    $C(s,a) \leftarrow 0$
    $\pi(s) \leftarrow \arg\max_a Q(s,a)$    (w

Loop forever (for each episode):
    $b \leftarrow$ any soft policy
    Generate an episode using $b$:
    $G \leftarrow 0$
    $W \leftarrow 1$
    Loop for each step of episode, $t = T-1, T-2, \ldots, 0$:
        $G \leftarrow \gamma G + R_{t+1}$
        $C(S_t, A_t) \leftarrow C(S_t, A_t) + W$
        $Q(S_t, A_t) \leftarrow \boxed{Q(S_t, A_t) + \frac{W}{C(S_t, A_t)}\left[G - Q(S_t, A_t)\right]}$
        $\pi(S_t) \leftarrow \arg\max_a Q(S_t, a)$    (with ties broken consistently)
        If $A_t \neq \pi(S_t)$ then exit inner Loop (proceed to next episode)
        $W \leftarrow W \frac{1}{b(A_t|S_t)}$

$$Q_n = \frac{\sum_{i=1}^{n} W_i G_i}{\sum_{i=1}^{n} W_i} = \frac{\sum_{i=1}^{n} W_i G_i}{C - W}$$

$$Q_{n+1} = \frac{\sum_{i=1}^{n+1} W_i G_i}{\sum_{i=1}^{n+1} W_i} = \frac{(C - W)Q_n + WG}{C}$$

$$= \frac{C}{C} Q_n - \frac{W}{C} Q_n + \frac{W}{C} G = Q_n + \frac{W}{C}\left[G - Q_n\right]$$

**Questions:**

1. Will this procedure converge to the optimal policy $\pi^*$?
2. Why do we break when $A_t \neq \pi(S_t)$?
3. Why do the weights $W$ not involve $\pi(A_t \mid S_t)$?

# Summary

- Estimating action values requires either **exploring starts** or a **soft policy** (e.g., $\epsilon$-greedy)
- **Off-policy learning** is the estimation of value functions for a **target policy** based on episodes generated by a different **behaviour policy**
    - **Importance sampling** is one way to perform off-policy learning
    - **Weighted** importance sampling has lower **variance** than **ordinary** importance sampling
- **Off-policy control** is learning the **optimal policy** (target policy) using episodes from a **behaviour policy**