Inference in Belief Networks

CMPUT 366: Intelligent Systems

P&M §8.4

Lecture Outline

1. Recap
2. Factors
3. Variable Elimination
4. Efficiency

Recap: Belief Networks

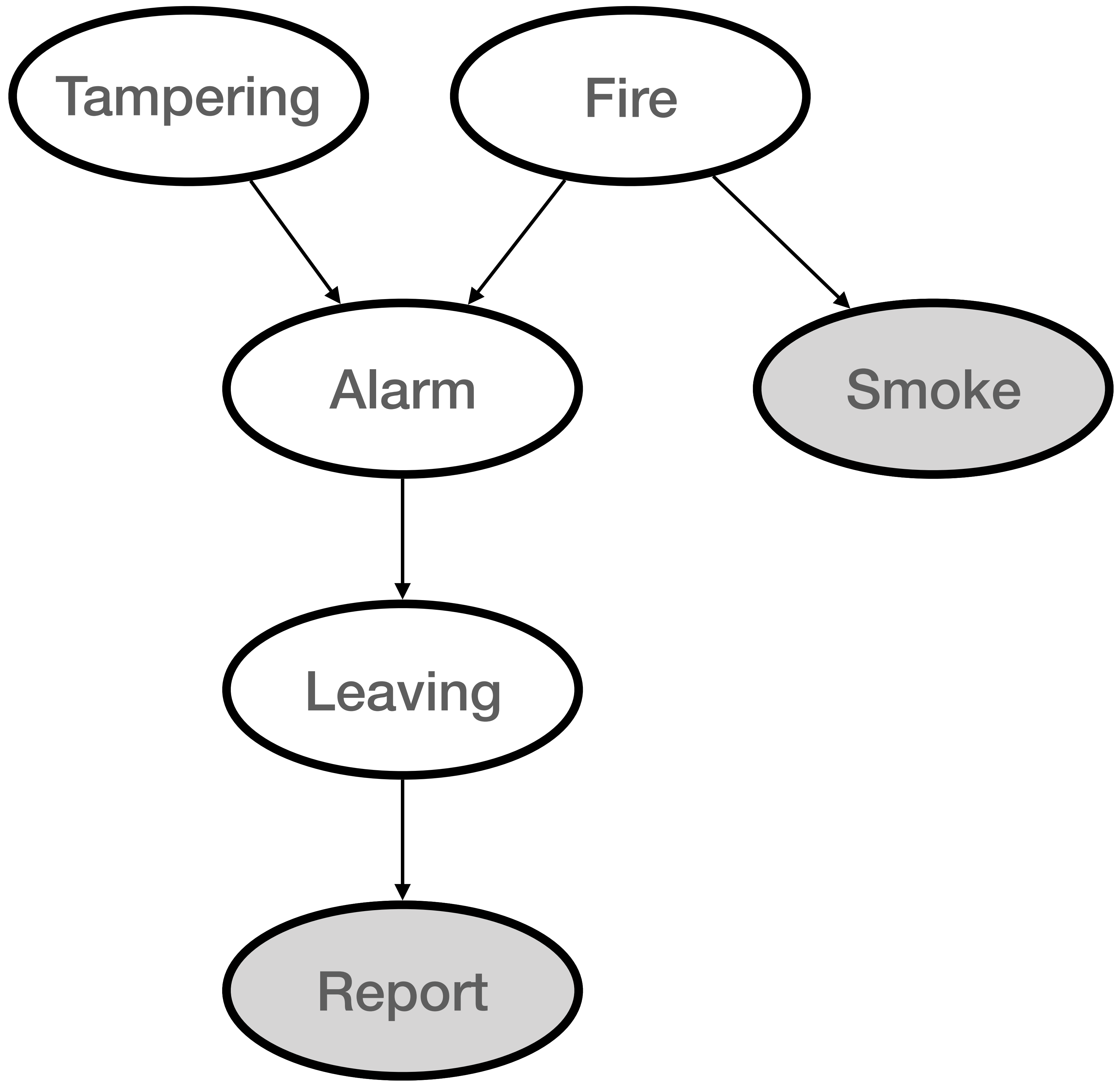
Definition:
A **belief network** (or **Bayesian network**) consists of:

1. A directed acyclic graph, with each node labelled by a **random variable**
2. A **domain** for each random variable
3. A **conditional probability table** for each variable given its **parents**

- The graph represents a specific **factorization** of the full **joint distribution**
- **Semantics:**
  Every node is **independent** of its **non-descendants**, **conditional** on its **parents**

Recap: Queries

- The most common task for a belief network is to query **posterior probabilities** given some **observations**
- **Easy cases:**
    - Posteriors of a **single variable** conditional only on **parents**
    - **Joint distributions** of variables early in a **compatible** **variable ordering**
- Typically, the observations have **no straightforward relationship** to the target
- This lecture: mechanical procedure for computing **arbitrary queries**



Factors

- The **Variable Elimination** algorithm exploits the **factorization** of a joint probability distribution encoded by a belief network in order to answer **queries**
- A **factor** is a function $f(X_1, \ldots, X_k)$ from **random variables** to a **real number**
- **Input:** factors representing the **conditional probability tables** from the belief network

$$P(Leaving \mid Alarm)P(Smoke \mid Fire)P(Alarm \mid Tampering, Fire)P(Tampering)P(Fire)$$

becomes

$$f_1(Leaving, Alarm)f_2(Smoke, Fire)f_3(Alarm, Tampering, Fire)f_4(Tampering)f_5(Fire)$$

- **Output:** A **new factor** encoding the target **posterior distribution**

E.g., $f_{12}(Tampering)$.

Conditional Probabilities as Factors

- A **conditional probability** $P(Y \mid X_1, \ldots, X_n)$ is a factor $f(Y, X_1, \ldots, X_n)$ that obeys the **constraint**:

$$\forall v_1 \in dom(X_1), v_2 \in dom(X_2), \ldots, v_n \in dom(X_n) : \left[\sum_{y \in dom(Y)} f(y, v_1, \ldots, v_n) \right] = 1.$$

- Answer to a query is a factor **constructed by applying operations** to the input factors
  - Operations on factors are not guaranteed to **maintain** this constraint!
  - Solution: **Don't sweat it**!
  - Operate on **unnormalized probabilities** during the computation
  - **Normalize** at the end of the algorithm to re-impose the constraint

Conditioning

Conditioning is an operation on a **single factor**

- Constructs a **new factor** that returns the values of the original factor with some of its inputs fixed

Definition:

For a factor $f_1(X_1, \ldots, X_k)$, **conditioning on** $X_i = v_i$ yields a new factor

$$f_2(X_1, \ldots X_{i-1}, X_{i+1}, \ldots, X_k) = (f_1)_{X_i = v_i}$$

such that for all values $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k$ in the domain of $X_1, \ldots X_{i-1}, X_{i+1}, \ldots, X_k$,

$$f_2(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k) = f_1(v_1, \ldots, v_{i-1}, \mathbf{v_i}, v_{i+1}, \ldots, v_k).$$

Conditioning Example

$$f_2(A,B) = f_1(A,B,C)_{C=true}$$

A	B	C	value
F	F	F	0.1
F	F	T	0.88
F	T	F	0.12
F	T	T	0.45
T	F	F	0.7
T	F	T	0.66
T	T	F	0.1
T	T	T	0.25

A	B	value
F	F	0.88
F	T	0.45
T	F	0.66
T	T	0.25

Multiplication

Multiplication is an operation on **two factors**

- Constructs a new factor that returns the **product** of the rows selected from each factor by its arguments

Definition:

For two factors $f_1(X_1, \ldots, X_j, Y_1, \ldots, Y_k)$ and $f_2(Y_1, \ldots, Y_k, Z_1, \ldots, Z_\ell)$,
multiplication of f_1 and f_2 yields a new factor

$$(f_1 \times f_2) = f_3(X_1, \ldots, X_j, Y_1, \ldots, Y_k, Z_1, \ldots, Z_\ell)$$

such that for all values $x_1, \ldots, x_j, y_1, \ldots, y_k, z_1, \ldots, z_\ell$,

$$f_3(x_1, \ldots, x_j, y_1, \ldots, y_k, z_1, \ldots, z_\ell) = f_1(x_1, \ldots, x_j, y_1, \ldots, y_k) f2(y_1, \ldots, y_k, z_1, \ldots, z_\ell).$$

Multiplication Example

$$f_3(A, B, C) = f_1(A, B) \times f_2(B, C)$$

A	B	value
F	F	0.1
F	T	0.2
T	F	0.3
T	T	0.4

B	C	value
F	F	1.0
F	T	0
T	F	0.5
T	T	0.25

A	B	C	value
F	F	F	0.1
F	F	T	0
F	T	F	0.1
F	T	T	0.05
T	F	F	0.3
T	F	T	0
T	T	F	0.2
T	T	T	0.1

Summing Out

Summing out is an operation on a **single factor**

- Constructs a new factor that returns the **sum over all values** of a random variable of the original factor

Definition:
For a factor $f_1(X_1, \ldots, X_k)$, summing out a variable X_i yields a new factor

$$f_2(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_k) = \left(\sum_{X_i} f_1 \right)$$

such that for all values $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k$ in the domain of $X_1, \ldots X_{i-1}, X_{i+1}, \ldots, X_k,$

$$f_2(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k) = \sum_{\mathbf{v_i} \in dom(X_i)} (v_1, \ldots, v_{i-1}, \mathbf{v_i}, v_{i+1}, \ldots, v_k).$$

Summing Out Example

$$f_2(B) = \sum_A f_1(A, B)$$

A	B	value
F	F	0.1
F	T	0.2
T	F	0.3
T	T	0.4

B	value
F	0.4
T	0.6

Variable Elimination

- Given **observations** $Y_1 = v_1, \ldots, Y_k = v_k$ and query variable Q, we want

$$P(Q \mid Y_1 = v_1, \ldots, Y_k = v_k) = \frac{P(Q, Y_1 = v_1, \ldots, Y_k = v_k)}{\sum_{q \in dom(Q)} P(Q = q, Y_1 = v_1, \ldots, Y_k = v_k)}.$$

- Basic idea of variable elimination:
  1. Condition on observations by **conditioning**
  2. Construct joint distribution factor by **multiplication**
  3. Remove unwanted variables (neither query nor observed) by **summing out**
  4. **Normalize** at the end
- Doing these steps in order is **correct** but not **efficient**
- Efficiency comes from **interleaving** the order of operations

Sums of Products

2. Construct joint distribution factor by **multiplication**
3. Remove unwanted variables (neither query nor observed) by **summing out**

The computationally intensive part of variable elimination is computing **sums** of **products**

Example: multiply factors $f_1(Q, A, B, C), f_2(C, D, E)$; sum out A, E

1. $f_3(Q, A, B, C, D, E) = f_1(Q, A, B, C) \times f_2(C, D, E)$: 2^6 multiplications
2. $f_4(Q, B, C, D) = \sum_{A,E} f_3(Q, A, B, C, D, E)$: 3×16 additions

Total: **112** computations

Efficient Sums of Products

We can reduce the number of computations required by changing their **order**.

$$\sum_A \sum_E f_1(Q,A,B,C) \times f_2(C,D,E)$$

$$= \left(\sum_A f_1(Q,A,B,C) \right) \times \left(\sum_E f_2(C,D,E) \right)$$

1. $f_3(C,D) = \Sigma_E f_2(C,D,E)$: 2^2 additions
2. $f_4(Q,B,C) = \Sigma_A f_1(Q,A,B,C)$: 2^3 additions
3. $f_5(Q,B,C,D) = f_3(Q,B,C) \times f_4(B,C,D)$: 2^4 multiplications

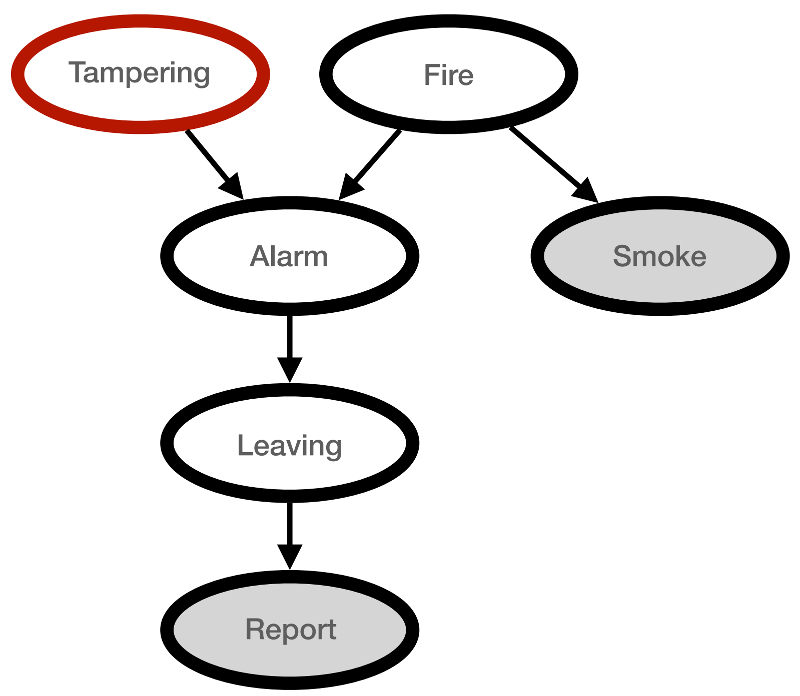
Total: **28** computations

Variable Elimination Algorithm

Input: query variable *Q*; set of variables *Vs*; observations *O*; factors *Ps* representing conditional probability tables

Fs := *Ps*
for each *X* in *Vs* \ {*Q*} according to some **elimination ordering**:
    Rs = { *F* in *Fs* | *F* involves *X* }
    if *X* is observed:
        for each *F* in *Rs*:
            F' = *F* **conditioned** on observed value of *X*
            Fs = *Fs* \ {*F*} ∪ {*F'*}
    else:
        T := **product** of factors in *Rs*
        N := **sum** *X* out of *T*
        Fs := *Fs* \ *Rs* ∪ {*N*}
T := **product** of factors in *Fs*
N := **sum** *Q* out of *T*
return *T* / *N*

Variable Elimination Example: Conditioning



Query: P(Tampering | Smoke=true, Report=true)
Variable ordering: Smoke, Report, Fire, Alarm, Leaving

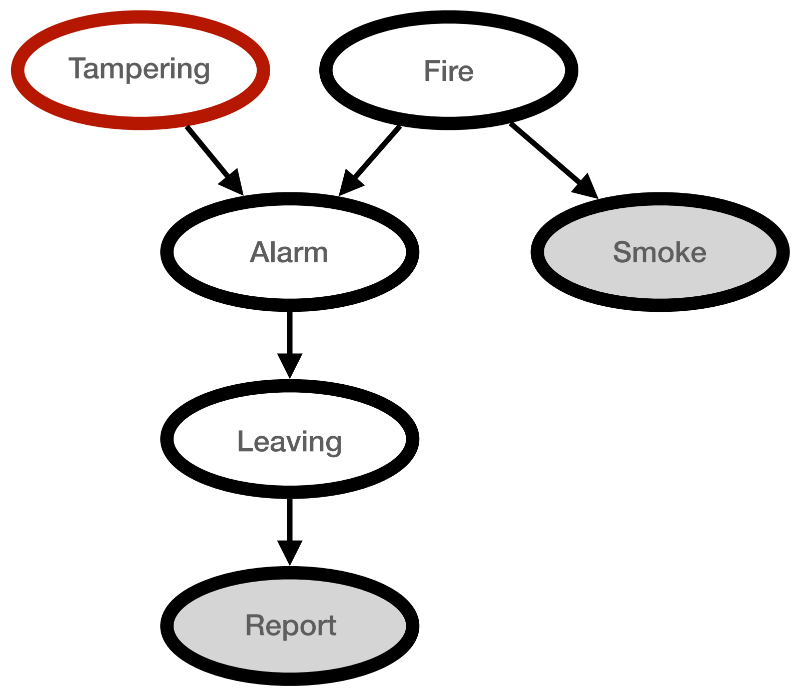
P(Tampering, Fire, Alarm, Smoke, Leaving, Report) =
P(Tampering)P(Fire)P(Alarm|Tampering,Fire)P(Smoke|Fire)P(Leaving|Alarm)P(Report|Leaving)

Construct **factors** for each table:
{ f_0(Tampering), f_1(Fire), f_2(Tampering,Alarm,Fire), f_3(Smoke,Fire), f_4(Leaving,Alarm), f_5(Report,Leaving) }

Condition on Smoke: $f_6 = (f_3)_{\text{Smoke=true}}$
{ f_0(Tampering), f_1(Fire), f_2(Tampering,Alarm,Fire), f_6(Fire), f_4(Leaving,Alarm), f_5(Report,Leaving) }

Condition on Report: $f_7 = (f_5)_{\text{Report=true}}$
{ f_0(Tampering), f_1(Fire), f_2(Tampering,Alarm,Fire), f_6(Fire), f_4(Leaving,Alarm), f_7(Leaving) }

Variable Elimination Example: Elimination



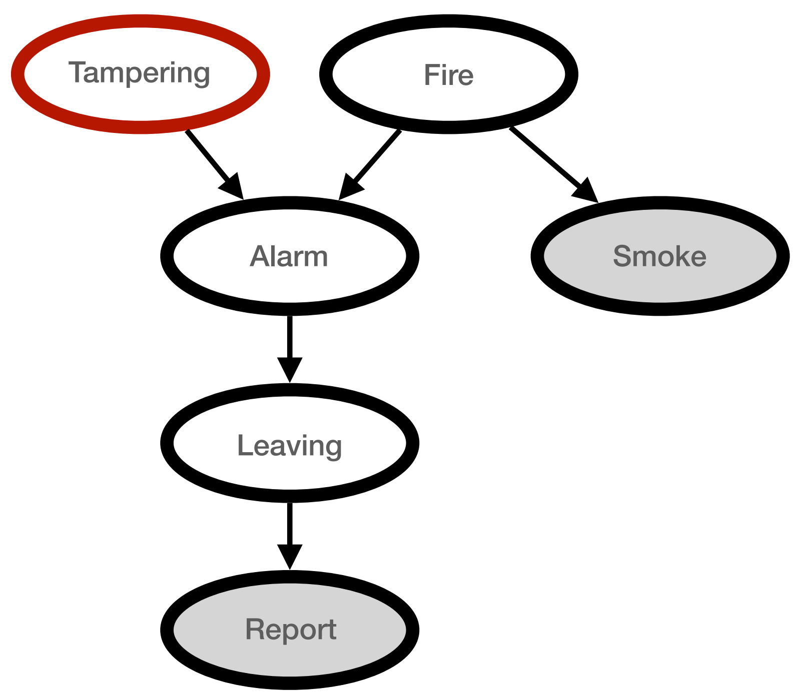
Query: P(Tampering | Smoke=true, Report=true)
Variable ordering: ~~Smoke, Report~~, Fire, Alarm, Leaving
{ f_0(Tampering), f_1(Fire), f_2(Tampering,Alarm,Fire), f_6(Fire), f_4(Leaving,Alarm), f_7(Leaving) }

Sum out Fire from **product** of f_1,f_2,f_6: $f_8 = \sum_{\text{Fire}} (f_1 \times f_2 \times f_6)$
{ f_0(Tampering), f_8(Tampering,Alarm), f_4(Leaving,Alarm), f_7(Leaving) }

Sum out Alarm from **product** of f_8, f_4: $f_9 = \sum_{\text{Alarm}} (f_8 \times f_4)$
{ f_0(Tampering), f_9(Tampering,Leaving), f_7(Leaving) }

Sum out Leaving from **product** of f_9, f_7: $f_{10} = \sum_{\text{Leaving}} (f_9 \times f_7)$
{ f_0(Tampering), f_{10}(Tampering) }

Variable Elimination Example: Normalization



Query: P(Tampering | Smoke=true, Report=true)
Variable ordering: ~~Smoke, Report, Fire, Alarm, Leaving~~
{ f_0(Tampering), f_{10}(Tampering) }

Product of remaining factors: $f_{11} = f_0 \times f_{10}$
{ f_{11}(Tampering) }

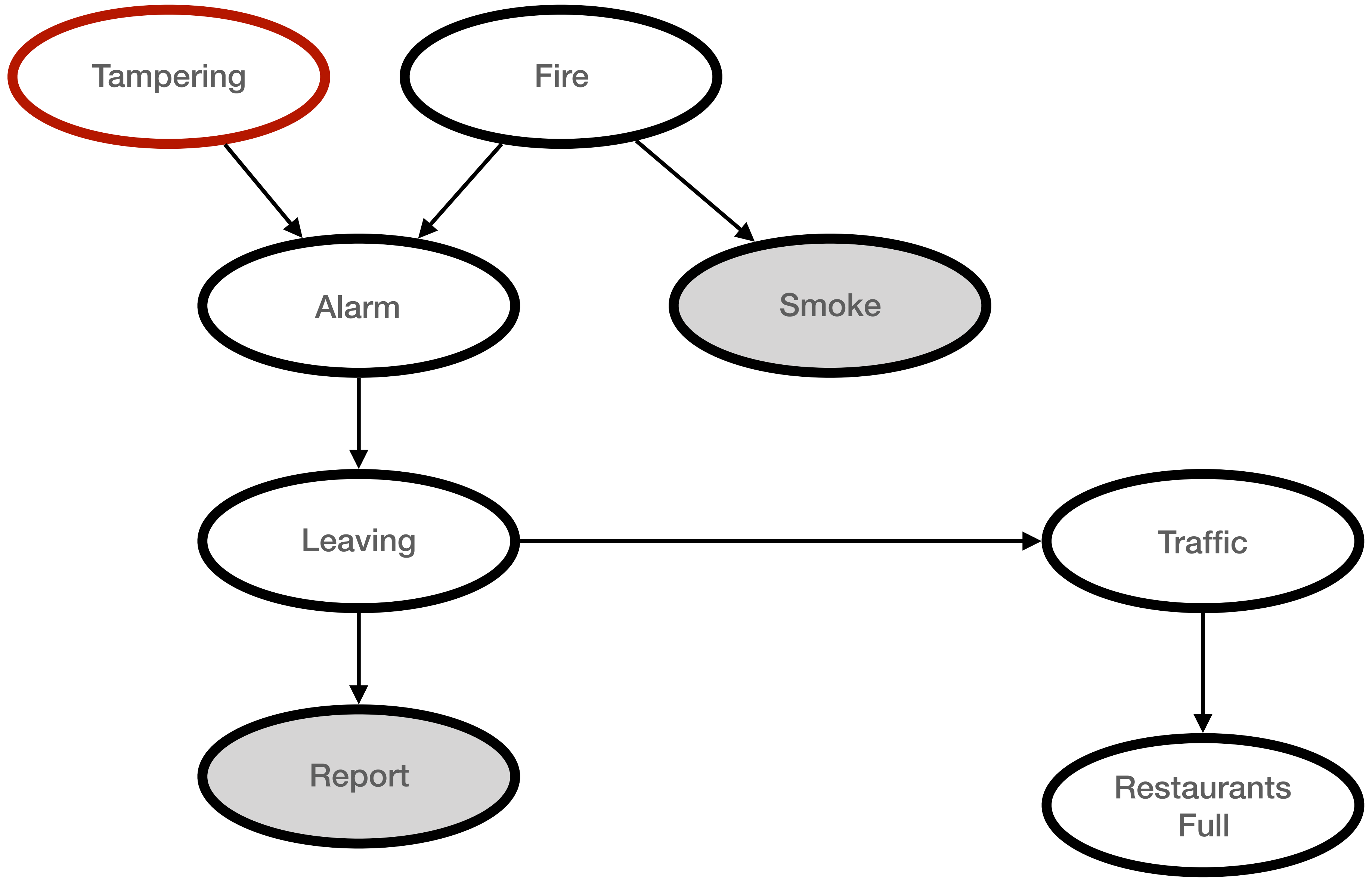
Normalize by division:
query(Tampering) = f_{11}(Tampering) / ($\sum_{\text{Tampering}} f_{11}$(Tampering))

Optimizing Elimination Order

- Variable elimination exploits **efficient sums of products** on a **factored joint distribution**
- The **elimination order** of the variables affects the **efficiency** of the algorithm
- Finding an **optimal** elimination ordering is **NP-hard**
- **Heuristics** (rules of thumb) for good orderings:
  - **Min-factor:** At every stage, select the variable that constructs the **smallest new factor**
  - **Problem-specific** heuristics

Optimization: Pruning

- The structure of the graph can allow us to **drop leaf nodes** that are **neither observed nor queried**
  - Summing them out for **free**
- We can **repeat** this process:



Optimization: Preprocessing

Finally, if we know that we are always going to be observing and/or querying the same variables, we can **preprocess** our graph; e.g.:

1. Precompute the **joint distribution** of all the variables we will observe and/or query
2. Precompute **conditional distributions** for our exact queries

Summary

- **Variable elimination** is an algorithm for answering **queries** based on a **belief network**
- Operates by using three **operations** on **factors** to reduce graph to a single posterior distribution
    1. Conditioning
    2. Multiplication
    3. Summing out
- **Distributes** operations more efficiently than taking full product and then summing out
    - **Optimal** order of operations is **NP-hard** to compute
- Additional **optimization** techniques: heuristic ordering, pruning, precomputation