Game Theory for Sequential Interactions

CMPUT 366: Intelligent Systems

S&LB §5.0-5.2.2

Lecture Outline

1. Recap
2. Perfect Information Games
3. Backward Induction
4. Imperfect Information Games

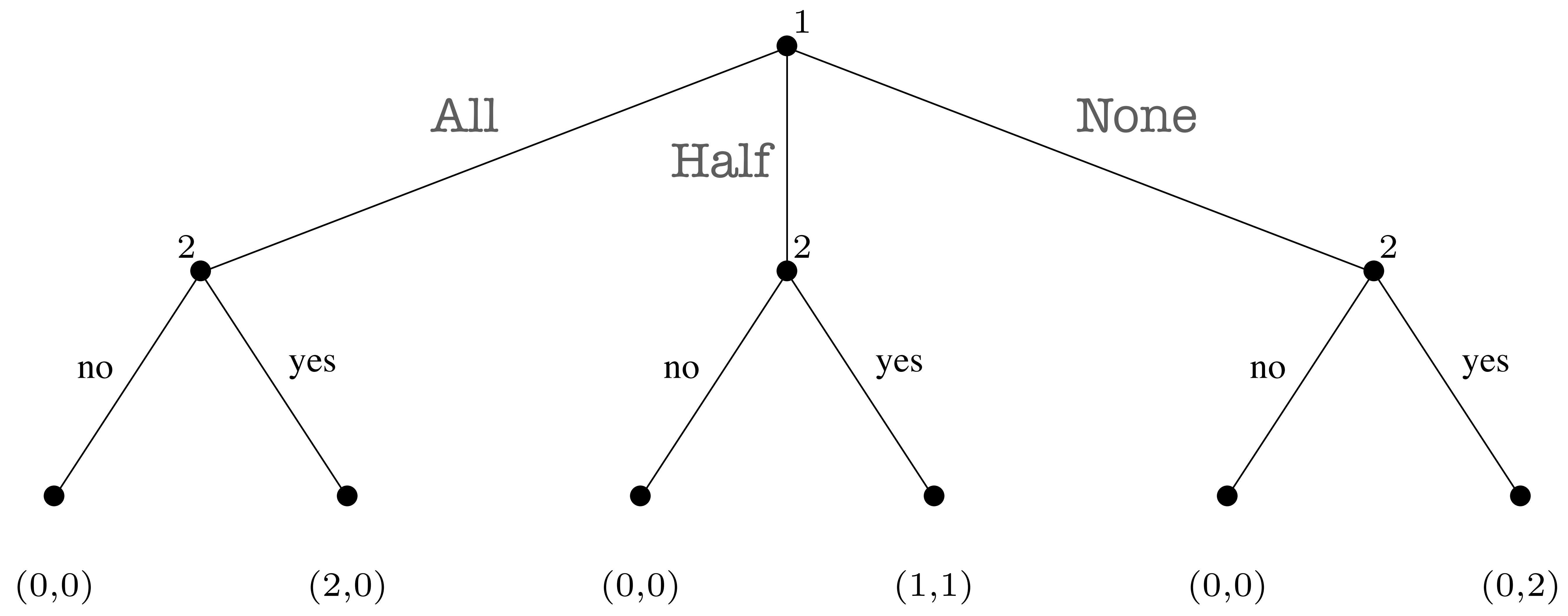
Recap: Game Theory

- Game theory studies the **interactions of rational agents**
    - Canonical representation is the **normal form game**
- Game theory uses **solution concepts** rather than optimal behaviour
    - "Optimal behaviour" is not clear-cut in multiagent settings
    - **Pareto optimal**: no agent can be made better off without making some other agent worse off
    - **Nash equilibrium**: no agent regrets their strategy given the choice of the other agents' strategies

	Ballet	Soccer
Ballet	2, 1	0, 0
Soccer	0, 0	1, 2

Extensive Form Games

- Normal form games don't have any notion of **sequence**: all actions happen **simultaneously**
- The **extensive form** is a game representation that explicitly includes temporal structure (i.e., a **game tree**)



Perfect Information

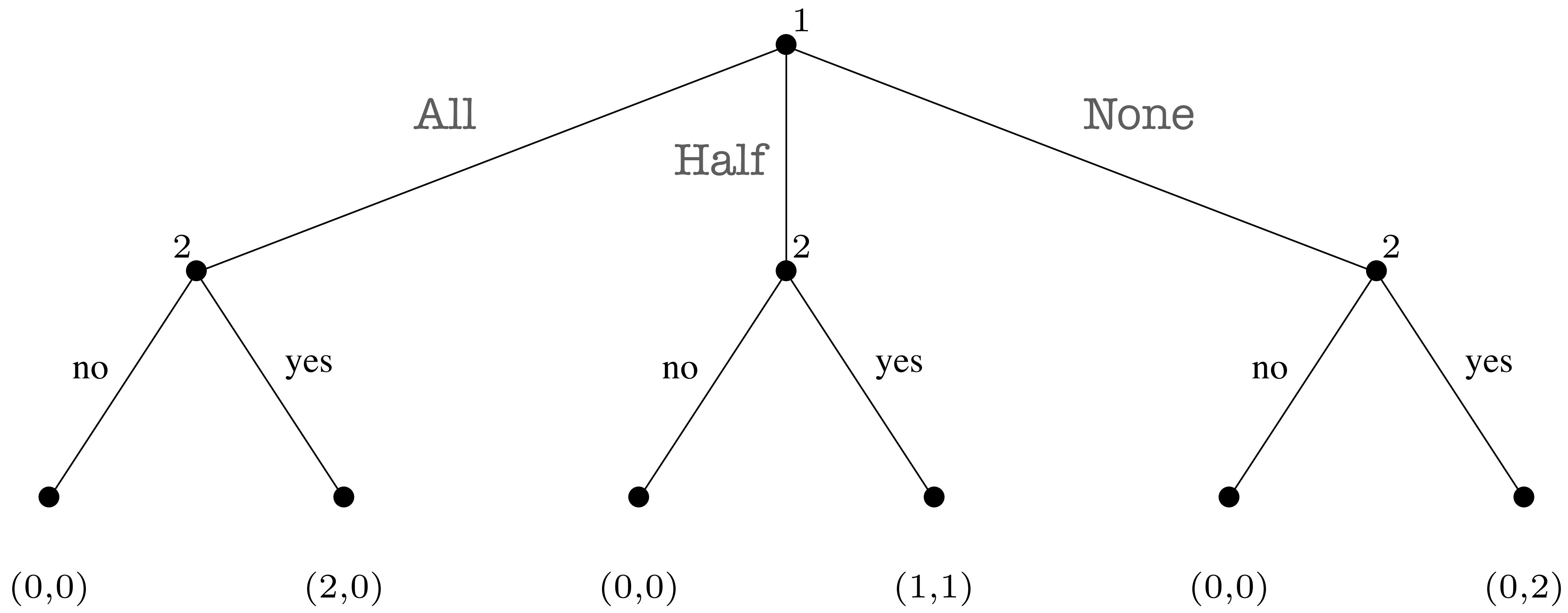
There are two kinds of extensive form game:

1. **Perfect information:** Every agent **sees all actions** of the other players (including Nature)
   - e.g.: Chess, checkers, Pandemic
2. **Imperfect information:** Some actions are **hidden**
   - Players may not know exactly where they are in the tree
   - e.g.: Poker, rummy, Scrabble

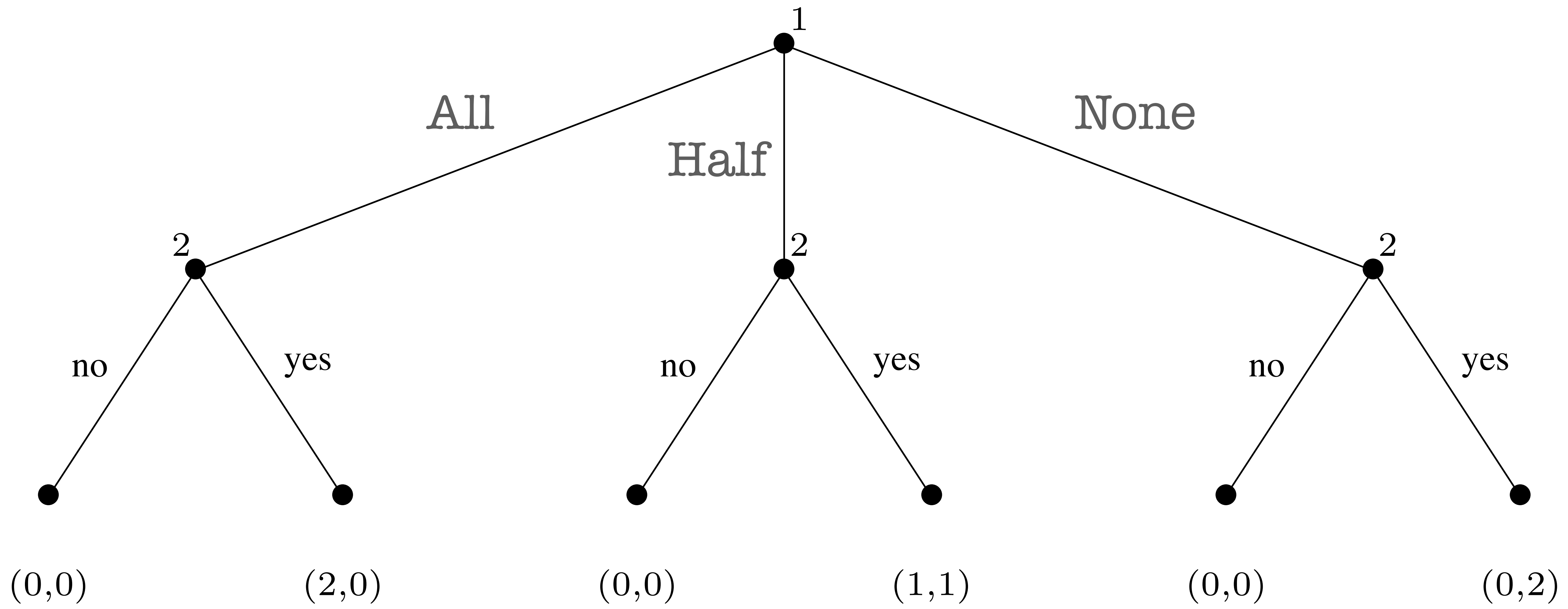
Perfect Information Extensive Form Game

Definition:
A **finite perfect-information game in extensive form** is a tuple $G = (N, A, H, Z, \chi, \rho, \sigma, u)$, where

- N is a set of n **players**,
- A is a single set of **actions**,
- H is a set of nonterminal **choice nodes**,
- Z is a set of **terminal nodes** (disjoint from H),
- $\chi : H \to 2^A$ is the **action function**,
- $\rho : H \to N$ is the **player function**,
- $\sigma : H \times A \to H \cup Z$ is the **successor function**,
- $u = (u_1, u_2, \ldots, u_n)$ is a **utility function** for each player, $u_i : Z \to \mathbb{R}$



Fun Game:
The Sharing Game



- Two siblings must decide how to share two $100 coins
- Sibling 1 suggests a division, then sibling 2 accepts or rejects
    - If rejected, nobody gets any coins.

Pure Strategies

Question: What are the **pure strategies** in an extensive form game?

Definition:
Let $G = (N, A, H, Z, \chi, \rho, \sigma, u)$ be a perfect information game in extensive form. Then the **pure strategies of player *i*** consist of the cross product of actions available to player *i* at each of their choice nodes, i.e.,

$$\prod_{h \in H \mid \rho(h) = i} \chi(h)$$

- A pure strategy associates an action with **each** choice node, even those that will **never be reached**

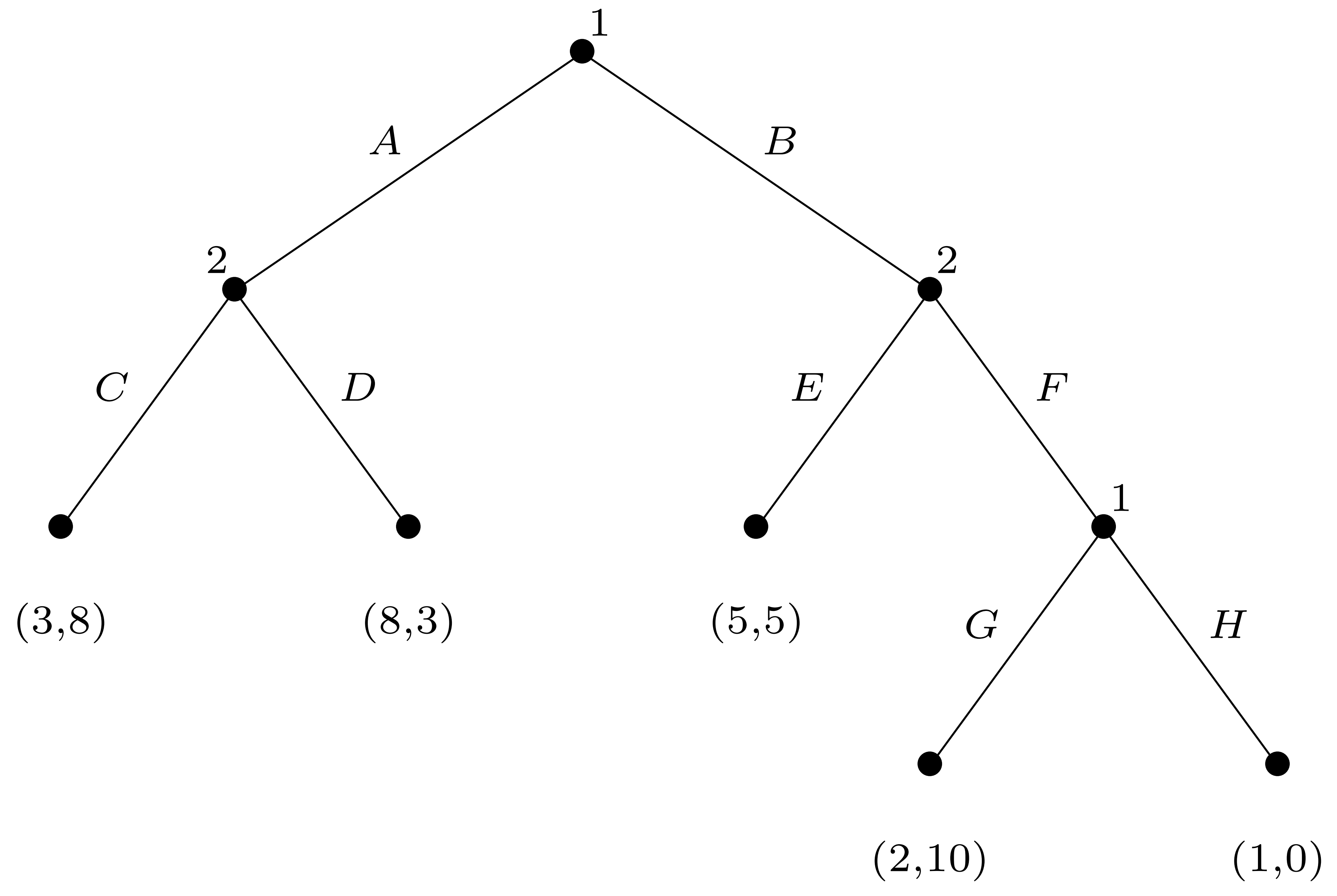
Pure Strategies Example

Question: What are the **pure strategies** for **player 2**?

- {(C,E), (C,F), (D,E), (D,F)}

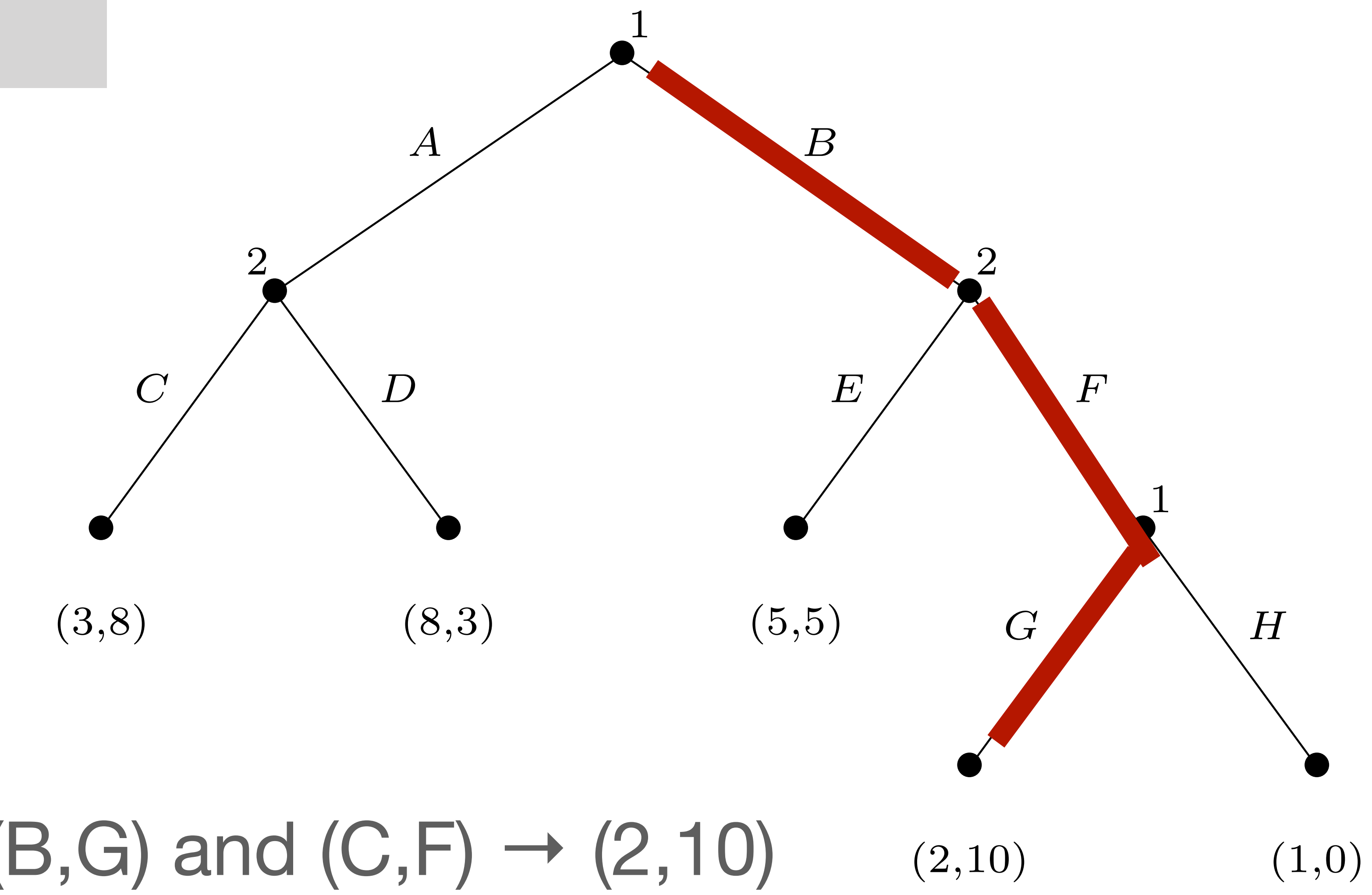
Question: What are the **pure strategies** for **player 1**?

- {(A,G), (A,H), (B,G), (B,H)}
- Note that these associate an action with the second choice node even when it can never be reached



Induced Normal Form

Question:
Which representation is more **compact**?



(B,G) and (C,F) → (2,10)

	C,E	C,F	D,E	D,F
A,G	3,8	3,8	8,3	8,3
A,H	3,8	3,8	8,3	8,3
B,G	5,5	2,10	5,5	2,10
B,H	5,5	1,0	5,5	1,0

- Any pair of pure strategies uniquely identifies a **terminal node**, which identifies a **utility** for each agent
- We have now defined a set of **agents**, **pure strategies**, and **utility functions**
- Any extensive form game defines a corresponding **induced normal form game**

Reusing Old Definitions

- We can plug our new definition of **pure strategy** into our existing definitions for:
  - Mixed strategy
  - Best response
  - Nash equilibrium (both pure and mixed strategy)

Question:

What is the definition of a **mixed strategy** in an extensive form game?

Pure Strategy Nash Equilibria

Theorem: [Zermelo, 1913]
Every finite perfect-information game in extensive form has at least one **pure strategy Nash equilibrium**.

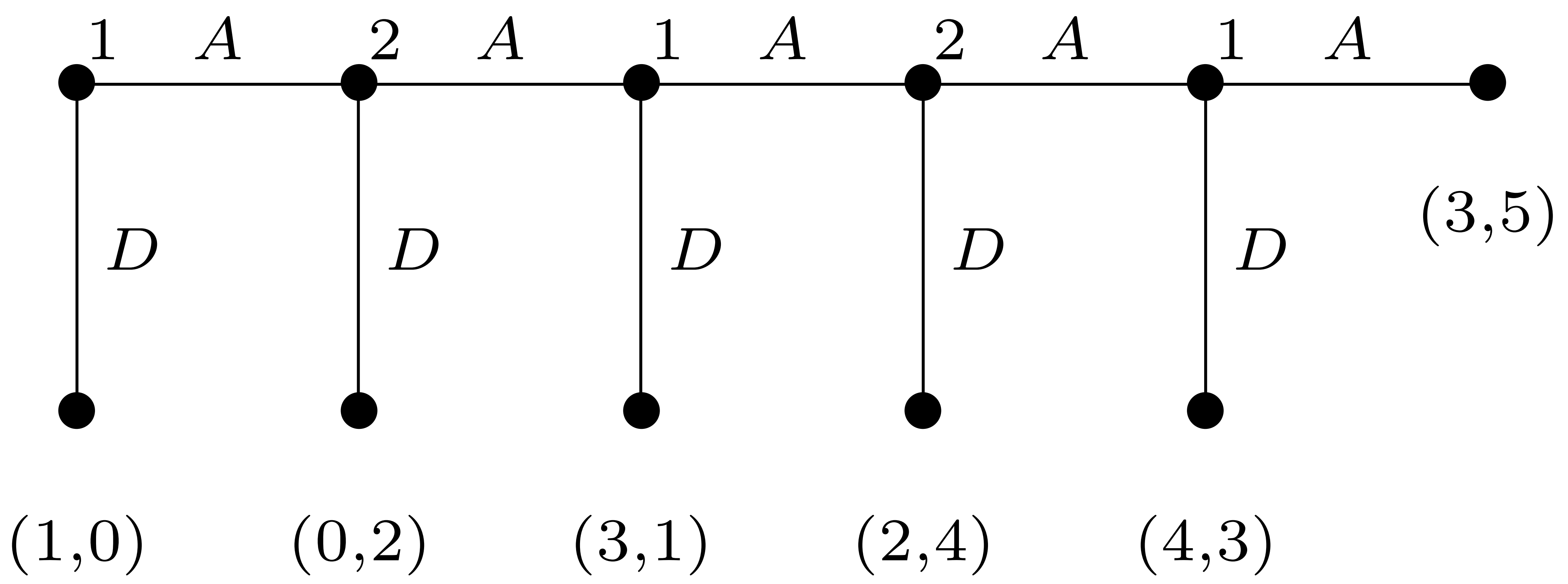
- Starting from the bottom of the tree, no agent needs to **randomize**, because they already know the best response
- There might be **multiple** pure strategy Nash equilibria in cases where an agent has multiple best responses at a **single choice node**

Backward Induction

- **Backward induction** is an algorithm for computing a pure strategy equilibrium in a perfect-information extensive-form game.
- **Idea:** Replace **subgames** in the tree with their **equilibrium values**

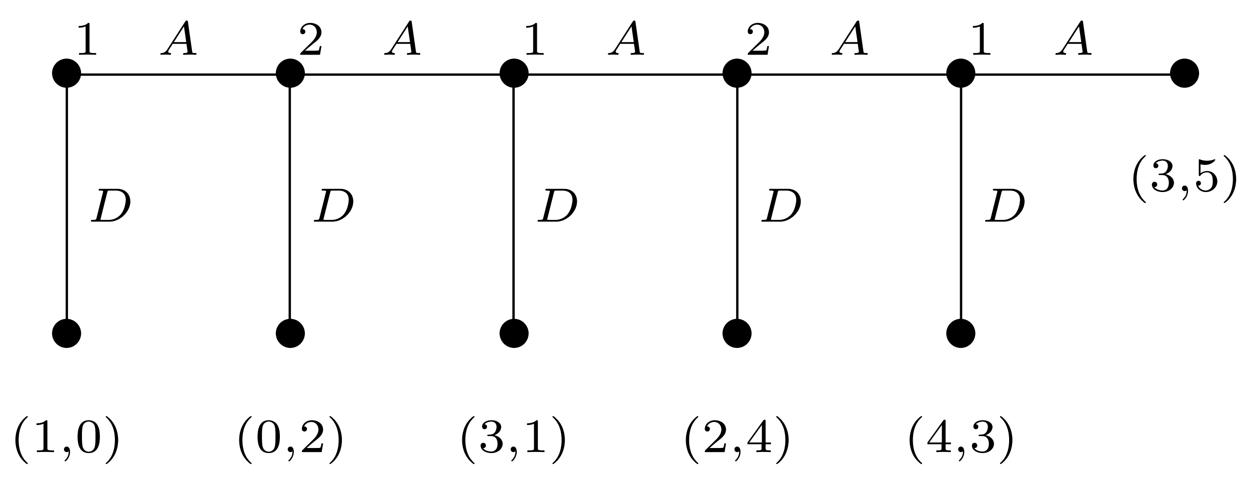
```
BACKWARDINDUCTION(h):
    if h is terminal:
        return u(h)
    i := ρ(h)
    U := -∞
    for each h' in χ(h):
        V = BACKWARDINDUCTION(h')
        if V_i > U_i:
            U_i := V_i
    return U
```

Fun Game: Centipede



- At each stage, one of the players can go `Across` or `Down`
- If they go `Down`, the game ends.

Backward Induction Criticism



- The **unique** equilibrium is for each player to go `Down` **at the first opportunity**
- **Empirically**, this is not how real people tend to play!
- **Theoretically**, what should you do if you arrive at an **off-path** node?
  - How do you update your beliefs to account for this probability 0 event?
  - If player 1 knows that you will update your beliefs in a way that causes you not to go `down`, then going `down` is no longer their only rational choice...

Imperfect Information, informally

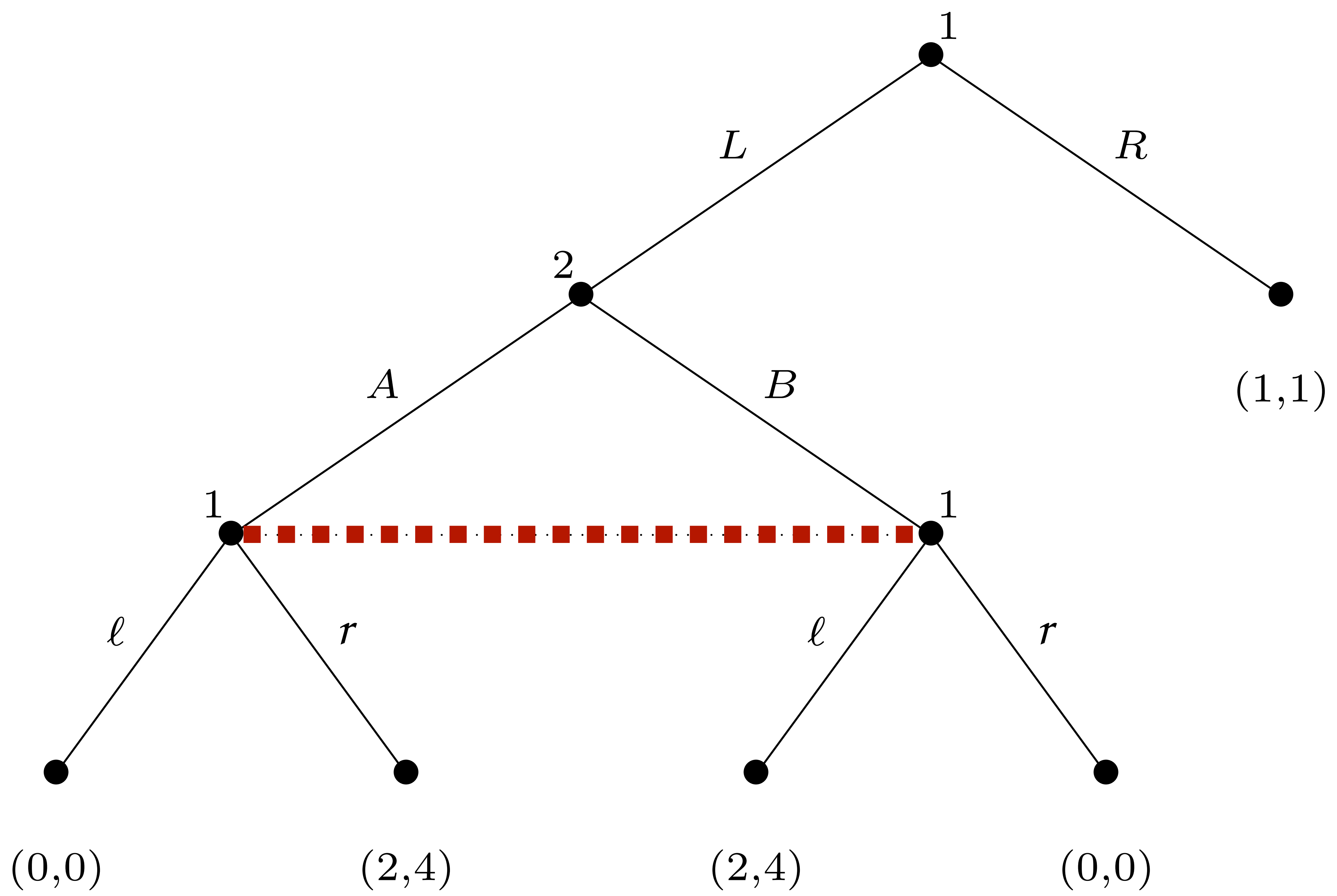
- **Perfect information** games model **sequential** actions that are **observed by all players**
  - **Randomness** can be modelled by a special *Nature* player with constant utility and known mixed strategy
- But many games involve **hidden** actions
  - Cribbage, poker, Scrabble
  - Sometimes actions of the **players** are hidden, sometimes ***Nature*'s** actions are hidden, sometimes both
- **Imperfect information extensive form games** are a model of games with sequential actions, some of which may be **hidden**

Imperfect Information Extensive Form Game

Definition:
An **imperfect information game in extensive form** is a tuple $G = (N, A, H, Z, \chi, \rho, \sigma, u, I)$, where

- $(N, A, H, Z, \chi, \rho, \sigma, u)$ is a perfect information extensive form game, and
- $I = (I_1, \ldots, I_n)$, where $I_i = (I_{i,1}, \ldots, I_{i,k_i})$ is an **equivalence relation** on (i.e., partition of) $\{h \in H : \rho(h) = i\}$ with the property that $\chi(h) = \chi(h')$ and $\rho(h) = \rho(h')$ whenever there exists a *j* for which $h \in I_{i,j}$ and $h' \in I_{i,j}$.

Imperfect Information Extensive Form Example



- The members of the equivalence classes are also called **information sets**
- Players **cannot distinguish** which **history** they are in within an information set
- **Question:** What are the information sets for each player in this game?

Pure Strategies

Question: What are the **pure strategies** in an **imperfect information** game?

Definition:
Let $G = (N, A, H, Z, \chi, \rho, \sigma, u, I)$ be an imperfect information game in extensive form. Then the **pure strategies of player *i*** consist of the cross product of actions available to player *i* at each of their **information sets**, i.e.,

$$\prod_{I_{i,j} \in I_i} \chi(h)$$

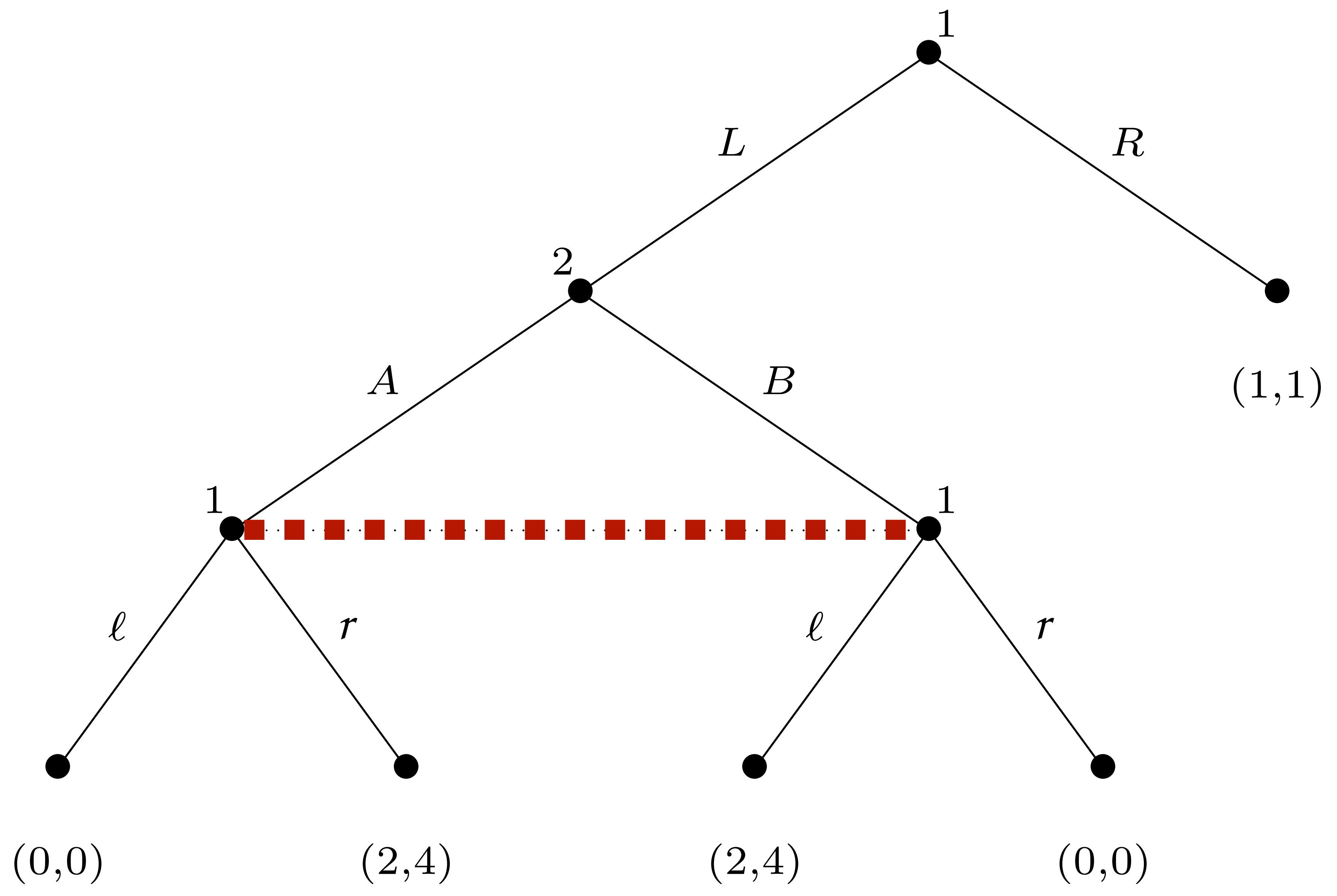
- A pure strategy associates an action with **each** information set, even those that will **never be reached**

Questions:

In an imperfect information game:

1. What are the **mixed strategies**?
2. What is a **best response**?
3. What is a **Nash equilibrium**?

Induced Normal Form



	A	B
L,ℓ	0,0	2,4
L,r	2,4	0,0
R,ℓ	1,1	1,1
R,r	1,1	1,1

Question:
Can you represent an arbitrary **perfect information** extensive form game as an **imperfect information** game?

- Any pair of pure strategies uniquely identifies a **terminal node**, which identifies a **utility** for each agent
- We have now defined a set of **agents**, **pure strategies**, and **utility functions**
- Any extensive form game defines a corresponding **induced normal form game**

Summary

- **Extensive form games** model **sequential** actions
- **Pure strategies** for extensive form games map **choice nodes** to **actions**
  - **Induced normal form:** normal form game with these pure strategies
  - Notions of mixed strategy, best response, etc. **translate directly**
- **Perfect information:** Every agent **sees all actions** of the other players
  - **Backward induction** computes a **pure strategy Nash equilibrium** for any perfect information extensive form game
- **Imperfect information:** Some actions are **hidden**
  - Histories are partitioned into **information sets**; players **cannot distinguish** between histories in the same information set