Policy Iteration & Monte Carlo Prediction

CMPUT 366: Intelligent Systems

S&B §4.3-4.4, 5.0-5.2

Lecture Outline

1. Recap
2. Policy Iteration
3. Monte Carlo Prediction

Recap: In-Place Iterative Policy Evaluation

Iterative Policy Evaluation, for estimating $V \approx v_\pi$

Input π, the policy to be evaluated
Algorithm parameter: a small threshold $\theta > 0$ determining accuracy of estimation
Initialize $V(s)$, for all $s \in \mathcal{S}^+$, arbitrarily except that $V(\mathit{terminal}) = 0$

Loop:
    $\Delta \leftarrow 0$
    Loop for each $s \in \mathcal{S}$:
        $v \leftarrow V(s)$
        $V(s) \leftarrow \sum_a \pi(a|s) \sum_{s',r} p(s',r|s,a)\big[r + \gamma V(s')\big]$
        $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$

- The updates are in-place: we use new values for V(s) immediately instead of waiting for the current sweep to complete
- These are **expected updates**: Based on a weighted average (expectation) of **all possible next states**

Recap:
Policy Improvement Theorem

Let π and π' be any pair of deterministic policies.

If $\quad q_\pi(s, \pi'(s)) \geq v_\pi(s) \quad \forall s \in \mathcal{S},$

then $\quad v_{\pi'}(s) \geq v_\pi(s) \quad \forall s \in \mathcal{S}.$

If you are never worse off at any state by following π' for **one step** and then following π forever after, then following π' **forever** has a higher expected value **at every state**

Policy Improvement Theorem Proof

$$
\begin{aligned}
v_\pi(s) &\leq q_\pi(s, \pi'(s)) \\
&= \mathbb{E}[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s, A_t = \pi'(s)] \\
&= \mathbb{E}_{\pi'}[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s] \\
&\leq \mathbb{E}_{\pi'}[R_{t+1} + \gamma q_\pi(S_{t+1}, \pi'(S_{t+1})) \mid S_t = s] \\
&= \mathbb{E}_{\pi'}[R_{t+1} + \gamma \mathbb{E}_{\pi'}[R_{t+2} + \gamma v_\pi(S_{t+2}) | S_{t+1}, A_{t+1} = \pi'(S_{t+1})] \mid S_t = s] \\
&= \mathbb{E}_{\pi'}\big[R_{t+1} + \gamma R_{t+2} + \gamma^2 v_\pi(S_{t+2}) \bigm| S_t = s\big] \\
&\leq \mathbb{E}_{\pi'}\big[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 v_\pi(S_{t+3}) \bigm| S_t = s\big] \\
&\ \ \vdots \\
&\leq \mathbb{E}_{\pi'}\big[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 R_{t+4} + \cdots \bigm| S_t = s\big] \\
&= v_{\pi'}(s).
\end{aligned}
$$

Greedy Policy Improvement

Given any policy π, we can construct a new greedy policy π' that is guaranteed to be **at least as good**:

$$\pi'(s) \doteq \arg\max_a q_\pi(s,a)$$
$$= \arg\max_a \mathbb{E}[R_{t+1} + \gamma v_\pi(S_{T+1}) \mid S_t = s, A_t = a]$$
$$= \arg\max_a \sum_{s',r} p(s',r \mid s,a)\Big[r + \gamma v_\pi(s')\Big] .$$

- If this new policy is **not better** than the old policy, then $v_\pi(s) = v_{\pi'}(s)$ for all s (**why?**) Because policy improvement theorem guarantees it is at least as good, so only way for it not to be better is to be the same.
- Also means that the new (and old) policies are **optimal** (**why?**)

If state values are the same after this update, then the Bellman optimality equation is satisfied, and v* is the unique solution to the Bellman optimal

Policy Iteration

$$\pi_0 \xrightarrow{\text{E}} v_{\pi_0} \xrightarrow{\text{I}} \pi_1 \xrightarrow{\text{E}} v_{\pi_1} \xrightarrow{\text{I}} \pi_2 \xrightarrow{\text{E}} \cdots \xrightarrow{\text{I}} \pi_* \xrightarrow{\text{E}} v_*$$

Policy Iteration (using iterative policy evaluation) for estimating $\pi \approx \pi_*$

1. Initialization
   $V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

2. Policy Evaluation
   Loop:
      $\Delta \leftarrow 0$
      Loop for each $s \in \mathcal{S}$:
         $v \leftarrow V(s)$
         $V(s) \leftarrow \sum_{s',r} p(s', r \,|\, s, \pi(s)) \big[r + \gamma V(s')\big]$
         $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
   until $\Delta < \theta$ (a small positive number determining the accuracy of estimation)

3. Policy Improvement
   policy-stable $\leftarrow$ *true*
   For each $s \in \mathcal{S}$:
      old-action $\leftarrow \pi(s)$
      $\pi(s) \leftarrow \arg\max_a \sum_{s',r} p(s', r \,|\, s, a) \big[r + \gamma V(s')\big]$
      If *old-action* $\neq \pi(s)$, then *policy-stable* $\leftarrow$ *false*
   If *policy-stable*, then stop and return $V \approx v_*$ and $\pi \approx \pi_*$; else go to 2

- This is a lot of iterations!
- Is it necessary to run to completion?

Value Iteration

Value iteration interleaves the estimation and improvement steps:

$$v_{k+1}(s) \doteq \max_a \mathbb{E}\left[R_{t+1} + \gamma v_k(S_{t+1}) \mid S_t = s, A_t = a\right]$$
$$= \max_a \sum_{s',r} p(s', r \mid s, a)\left[r + \gamma v_k(s')\right]$$

Value Iteration, for estimating $\pi \approx \pi_*$

Algorithm parameter: a small threshold $\theta > 0$ determining accuracy of estimation
Initialize $V(s)$, for all $s \in \mathcal{S}^+$, arbitrarily except that $V(\mathit{terminal}) = 0$

Loop:
| $\Delta \leftarrow 0$
| Loop for each $s \in \mathcal{S}$:
| $\quad v \leftarrow V(s)$
| $\quad V(s) \leftarrow \max_a \sum_{s',r} p(s', r \mid s, a)\left[r + \gamma V(s')\right]$
| $\quad \Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$

Output a deterministic policy, $\pi \approx \pi_*$, such that
$\pi(s) = \arg\max_a \sum_{s',r} p(s', r \mid s, a)\left[r + \gamma V(s')\right]$

Example: Blackjack

- Player gets two cards, dealer gets 1
- Player can `hit` (get a new card) as many times as they like, or `stick` (stop hitting)
- After the player is done, the dealer hits / sticks according to a fixed policy
- Whoever has the most points (sum of card values) wins
- But, if you have more than 21 points, you **lose immediately** ("bust")

Simulating Blackjack

- Given a policy for the player, it is **very easy** to simulate a game of Blackjack
- **Question:** Is it easy to compute the **full dynamics**?
- **Question:** Is it easy to run **iterative policy evaluation**?

Experience vs. Expectation

- In order to compute **expected updates**, we need to know the exact **probability** of **every** possible transition

- Often we don't have access to the full probability distribution, but we do have access to **samples of experience**

1. **Actual experience:** We want to learn based on interactions with a **real environment**, without knowing its dynamics

2. **Simulated experience:** We can **simulate** the dynamics, but we don't have an **explicit representation** of transition probabilities, or there are **too many states**

Monte Carlo Estimation

- **Question:** What was **Monte Carlo estimation** the last time we studied it (in Supervised Learning?)

- Instead of estimating expectations by a **weighted sum** over **all possibilities**, estimate expectation by **averaging** over a **sample** drawn from the distribution:

$$\mathbb{E}[X] = \sum_x f(x)x \approx \frac{1}{n}\sum_{i=1}^{n} x_i \quad \text{where } x_i \sim f$$

Monte Carlo Prediction

- Use a large **sample** of **episodes** generated by a policy π to estimate the state-values $v_\pi(s)$ for each state s
  - We will consider only **episodic** tasks for now
- **Question:** What is the **return** G_t for state S_t=s in a given episode?
- We can estimate the expected return $v_\pi(s) = \mathbb{E}[G_t \mid S_t{=}s]$ by averaging the returns for that state in every episode containing a visit to s

First-visit
Monte Carlo Prediction

First-visit MC prediction, for estimating $V \approx v_\pi$

Input: a policy π to be evaluated

Initialize:

$V(s) \in \mathbb{R}$, arbitrarily, for all $s \in \mathcal{S}$

$Returns(s) \leftarrow$ an empty list, for all $s \in \mathcal{S}$

Loop forever (for each episode):

Generate an episode following π: $S_0, A_0, R_1, S_1, A_1, R_2, \ldots, S_{T-1}, A_{T-1}, R_T$

$G \leftarrow 0$

Loop for each step of episode, $t = T-1, T-2, \ldots, 0$:

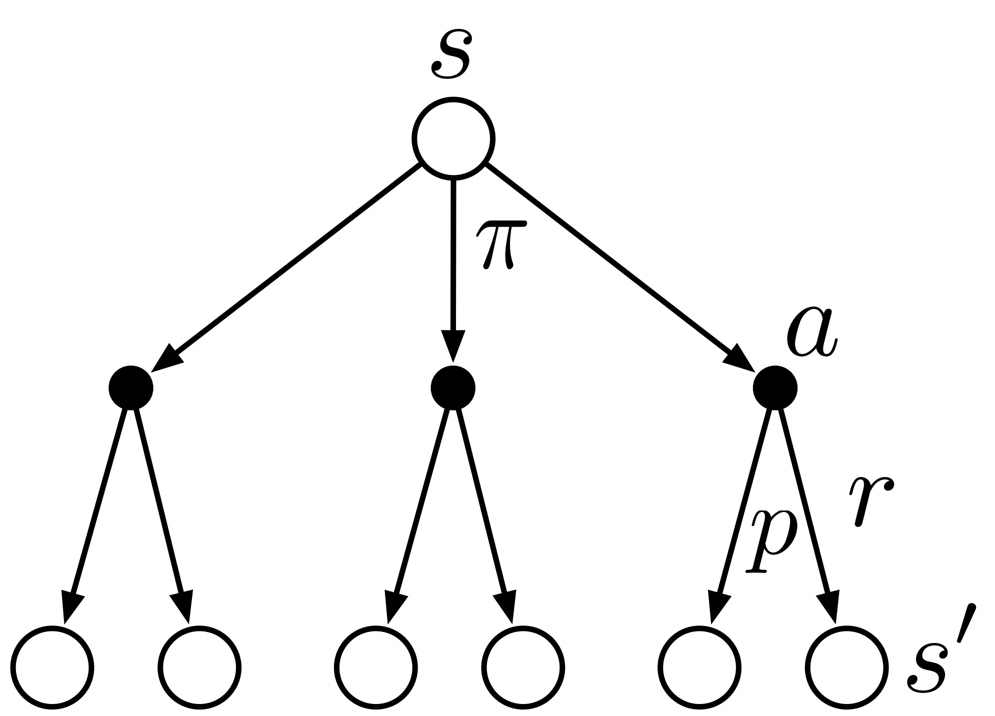
$G \leftarrow \gamma G + R_{t+1}$

Unless S_t appears in $S_0, S_1, \ldots, S_{t-1}$:

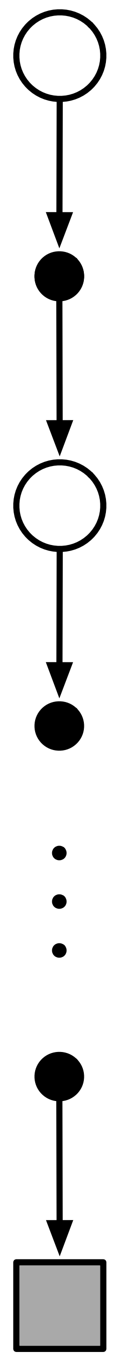
Append G to $Returns(S_t)$

$V(S_t) \leftarrow \text{average}(Returns(S_t))$

Monte Carlo vs. Dynamic Programming



- **Iterative policy evaluation** uses the estimates of the **next state's** value to update the value of this state
  - Only needs to compute a **single transition** to update a state's estimate
- **Monte Carlo** estimate of each state's value is **independent** from estimates of **other states'** values
  - Needs the **entire episode** to compute an update
  - Can focus on evaluating a **subset of states** if desired



Summary

- Given any policy π, we can compute a **greedy improvement** π' by choosing highest expected value action based on v_π
  - **Policy iteration:** Repeat:
    Greedy improvement using v_π, then recompute v_π
  - **Value iteration:** Repeat:
    Recompute v_π by assuming greedy improvement at every update
- **Monte Carlo estimation** estimates values by averaging returns over **sample episodes**
  - Does not require access to full model of dynamics