Optimality and Dynamic Programming

CMPUT 366: Intelligent Systems

S&B §3.6, §4.0-4.4

Lecture Outline

1. Assignment #3
2. Recap
3. Optimality
4. Policy Evaluation
5. Policy Improvement

Labs & Assignment #3

- Assignment #3 is due **Mar 25 (next Monday)** before lecture
- Today's lab is from **5:00pm to 7:50pm** in **CAB 235**
  - Not mandatory
  - Opportunity to get help from the TAs
- `mlp1` and `cnn` need to **train** and **evaluate** the specified models
  - **train:** fit parameters using provided training dataset
  - **evaluate:** compute loss on **both** provided test datasets

Recap: Value Functions

State-value function

$$v_\pi(s) \doteq \mathbb{E}_\pi[G_t \,|\, S_t = s]$$
$$= \mathbb{E}_\pi \left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \,\middle|\, S_t = s \right]$$

Action-value function

$$q_\pi(s, a) \doteq \mathbb{E}_\pi[G_t \,|\, S_t = s, A_t = a]$$
$$= \mathbb{E}_\pi \left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \,\middle|\, S_t = s, A_t = a \right]$$

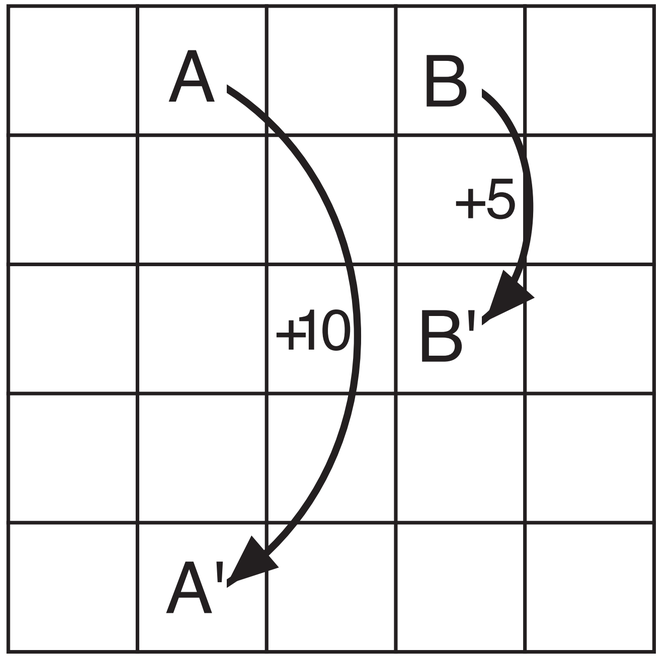
Recap: Bellman Equations

Value functions satisfy a **recursive consistency condition** called the **Bellman equation**:

$$
\begin{aligned}
v_\pi(s) &\doteq \mathbb{E}_\pi[G_t \,|\, S_t = s] \\
&= \mathbb{E}_\pi[R_{t+1} + \gamma G_{t+1} \,|\, S_t = s] \\
&= \sum_a \pi(a \,|\, s) \sum_{s'} \sum_r p(s', r \,|\, s, a) \Big[r + \gamma \mathbb{E}_\pi[G_{t+1} \,|\, S_{t+1} = s'] \Big] \\
&= \sum_a \pi(a \,|\, s) \sum_{s', r} p(s', r \,|\, s, a) \Big[r + \gamma v_\pi(s') \Big]
\end{aligned}
$$

- v_π is the **unique solution** to π's Bellman equation
- There is also a Bellman equation for π's **action-value function**

Recap: GridWorld Example



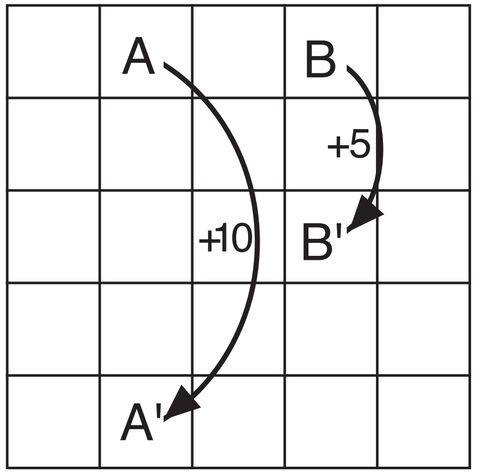
Reward dynamics

3.3	8.8	4.4	5.3	1.5
1.5	3.0	2.3	1.9	0.5
0.1	0.7	0.7	0.4	-0.4
-1.0	-0.4	-0.4	-0.6	-1.2
-1.9	-1.3	-1.2	-1.4	-2.0

State-value function v_π for random policy $\pi(a|s) = 0.25$

GridWorld with Bounds Checking

What about a policy where we never try to go over an edge?



Reward dynamics

3.3	8.8	4.4	5.3	1.5
1.5	3.0	2.3	1.9	0.5
0.1	0.7	0.7	0.4	-0.4
-1.0	-0.4	-0.4	-0.6	-1.2
-1.9	-1.3	-1.2	-1.4	-2.0

State-value function v_{π} for random policy $\pi(a|s) = 0.25$

6.7	**10.8**	**6.4**	**6.7**	**4.3**
4.2	**4.7**	**3.7**	**3.4**	**2.8**
2.4	**2.4**	**2.1**	**1.9**	**1.7**
1.5	**1.4**	**1.3**	**1.2**	**1.1**
1.1	**1.0**	**0.9**	**0.9**	**0.9**

State-value function v_{π_B} for **bounded** random policy π_B

Optimality

- **Question:** What is an **optimal** policy?
- A policy π is (weakly) **better** than a policy π' if it is better for all s $\in \mathcal{S}$:

$$\pi \geq \pi' \iff v_\pi(s) \geq v_{\pi'}(s) \quad \forall s \in \mathcal{S}$$

- An **optimal** policy π^* is weakly better than **every other policy**
- All optimal policies share the **same state-value function**: (**why?**)

$$v_*(s) \doteq \max_\pi v_\pi(s)$$

- Also the same **action-value function**:

$$q_*(s, a) \doteq \max_\pi q_\pi(s, a)$$

Bellman Optimality Equations

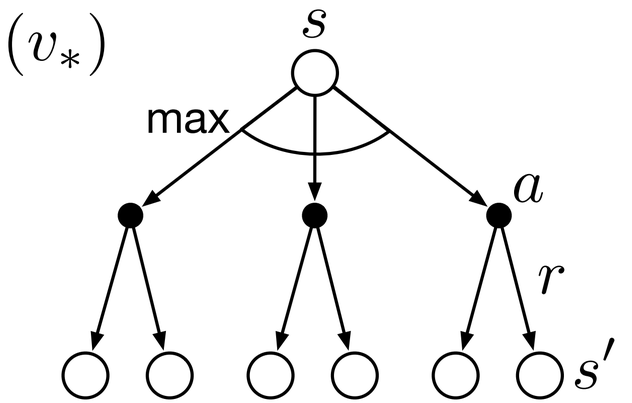
- v^* must satisfy the Bellman equation too

- In fact, it can be written in a special, **policy-free** way because we know that every state value is **maximized** by π^*:

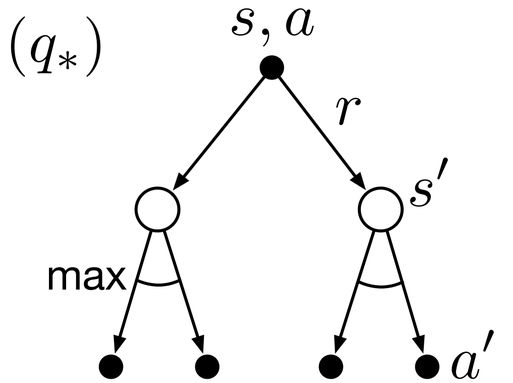
$$
\begin{aligned}
v_*(s) &= \max_a q_{\pi_*}(s,a) \\
&= \max_a \mathbb{E}_{\pi_*}[G_t \,|\, S_t = s, A_t = a] \\
&= \max_a \mathbb{E}_{\pi_*}[R_{t+1} + \gamma G_{t+1} \,|\, S_t = s, A_t = a] \\
&= \max_a \mathbb{E}[R_{t+1} + \gamma v_*(S_{t+1}) \,|\, S_t = s, A_t = a] \\
&= \max_a \sum_{s',r} p(s', r \,|\, s, a)[r + \gamma v_*(s')]
\end{aligned}
$$

Bellman Optimality Equations

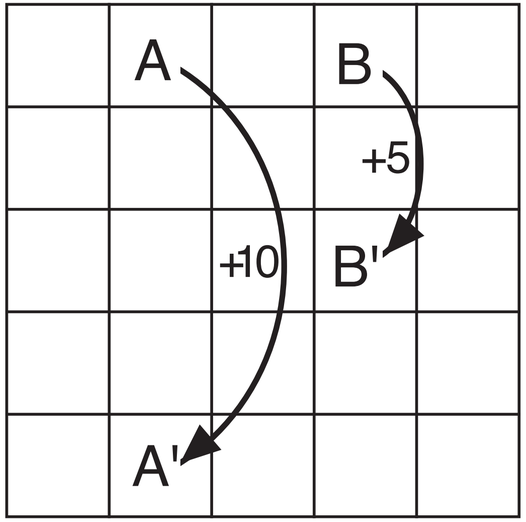
$$
\begin{aligned}
v_*(s) &= \max_a \mathbb{E}[R_{t+1} + \gamma v_*(S_{t+1}) \mid S_t = s, A_t = a] \\
&= \max_a \sum_{s',r} p(s', r \mid s, a)[r + \gamma v_*(s')]
\end{aligned}
$$



$$
\begin{aligned}
q_*(s, a) &= \mathbb{E}\left[R_{t+1} + \gamma \max_{a'} q_*(S_{t+1}, a') \,\middle|\, S_t = s, A_t = a\right] \\
&= \sum_{s',r} p(s', r \mid s, a)\left[r + \gamma \max_{a'} q_*(s', a')\right]
\end{aligned}
$$



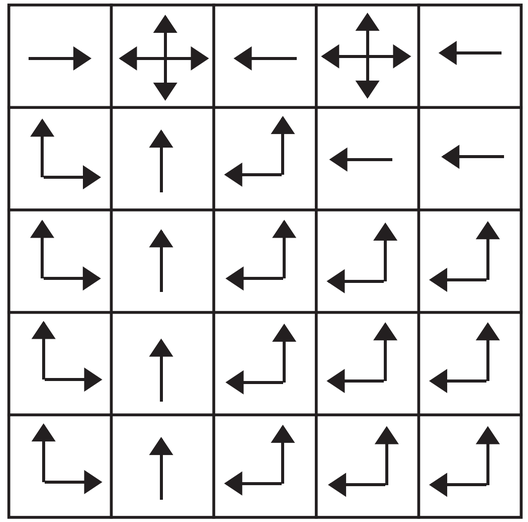
Optimal GridWorld



Gridworld

22.0	24.4	22.0	19.4	17.5
19.8	22.0	19.8	17.8	16.0
17.8	19.8	17.8	16.0	14.4
16.0	17.8	16.0	14.4	13.0
14.4	16.0	14.4	13.0	11.7

v_*



π_*

Policy Evaluation

Question: How can we **compute** v_π?

1. We know that v_π is the unique solution to the Bellman equations, so we could just solve them
    - but that is tedious and annoying and slow
    - Also requires a complete model of the dynamics
2. Iterative policy evaluation
    - Takes advantage of the recursive formulation

Iterative Policy Evaluation

- Iterative policy evaluation uses the Bellman equation as an **update rule**:

$$
\begin{aligned}
v_{k+1}(s) &\doteq \mathbb{E}_{\pi}[R_{t+1} + \gamma v_k(S_{t+1} \mid S_t = s] \\
&= \sum_a \pi(a \mid s) \sum_{s',r} p(s', r \mid s, a) \left[r + \gamma v_k(s') \right]
\end{aligned}
$$

- v_π is a **fixed point** of this update, by definition

- Furthermore, starting from an **arbitrary** v_0, the sequence $\{v_k\}$ will **converge** to v_π as $k \to \infty$

In-Place Iterative Policy Evaluation

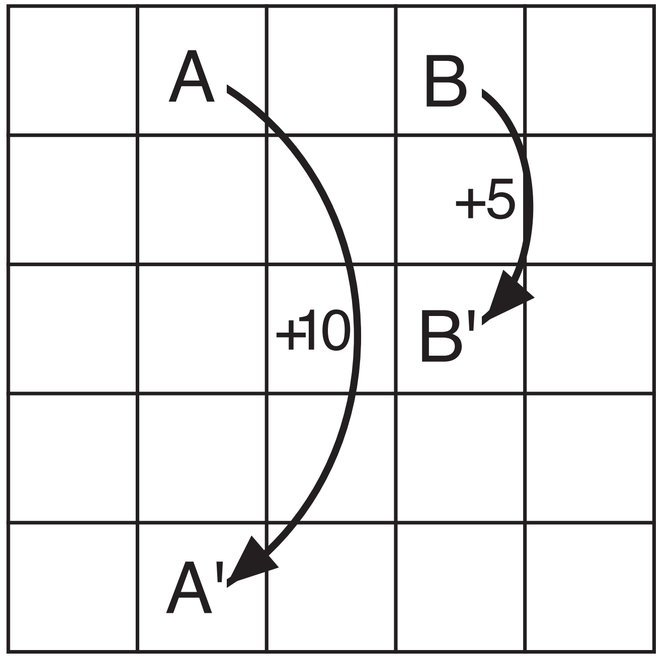
Iterative Policy Evaluation, for estimating $V \approx v_\pi$

Input π, the policy to be evaluated
Algorithm parameter: a small threshold $\theta > 0$ determining accuracy of estimation
Initialize $V(s)$, for all $s \in \mathcal{S}^+$, arbitrarily except that $V(terminal) = 0$

Loop:
  $\Delta \leftarrow 0$
  Loop for each $s \in \mathcal{S}$:
    $v \leftarrow V(s)$
    $V(s) \leftarrow \sum_a \pi(a|s) \sum_{s',r} p(s', r \,|\, s, a) \big[r + \gamma V(s')\big]$
    $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$

- The updates are in-place: we use new values for V(s) immediately instead of waiting for the current sweep to complete (**why?**)
- These are **expected updates**: Based on a weighted average (expectation) of **all possible next states** (**instead of what?**)

Iterative Policy Evaluation

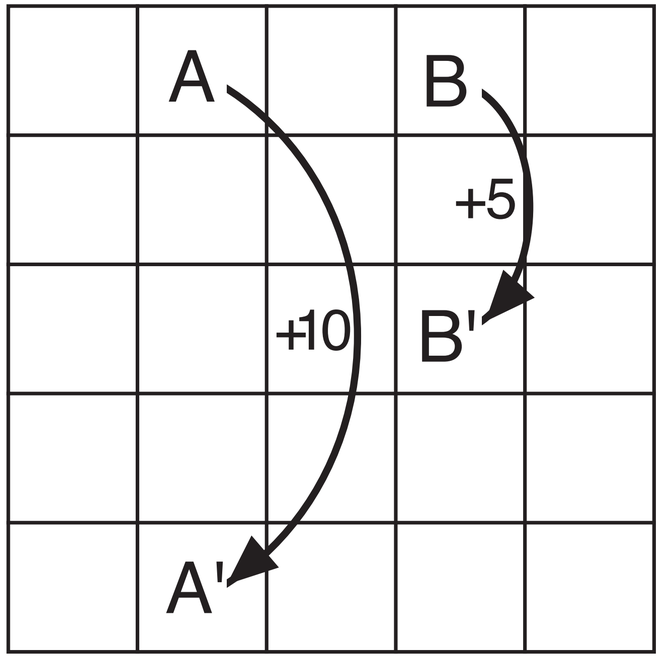


Reward dynamics

0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0

V at k=0

Iterative Policy Evaluation in GridWorld

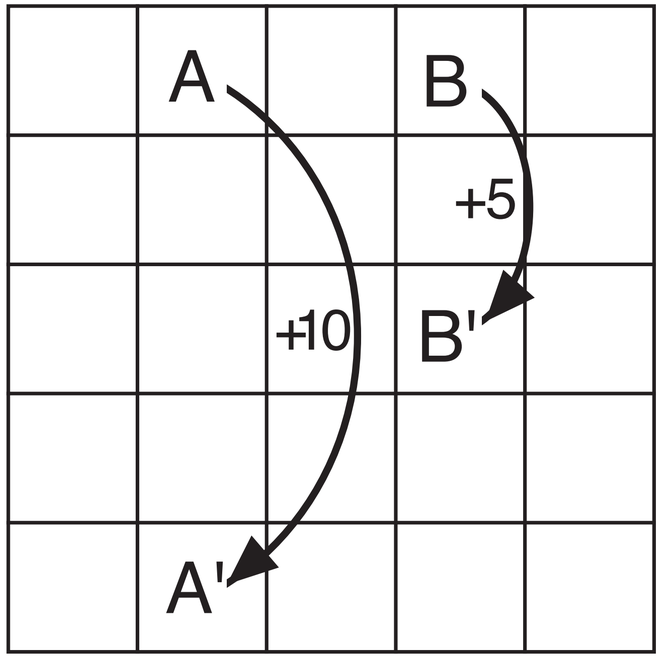


Reward dynamics

-0.5	10	2	5	0.6
-0.3	2.1	0.9	1.3	0.2
-0.3	0.4	0.3	0.4	-0.1
-0.3	0.0	0.0	0.1	-0.2
-0.5	-0.3	-0.3	-0.3	-0.6

V at k=1

Iterative Policy Evaluation in GridWorld

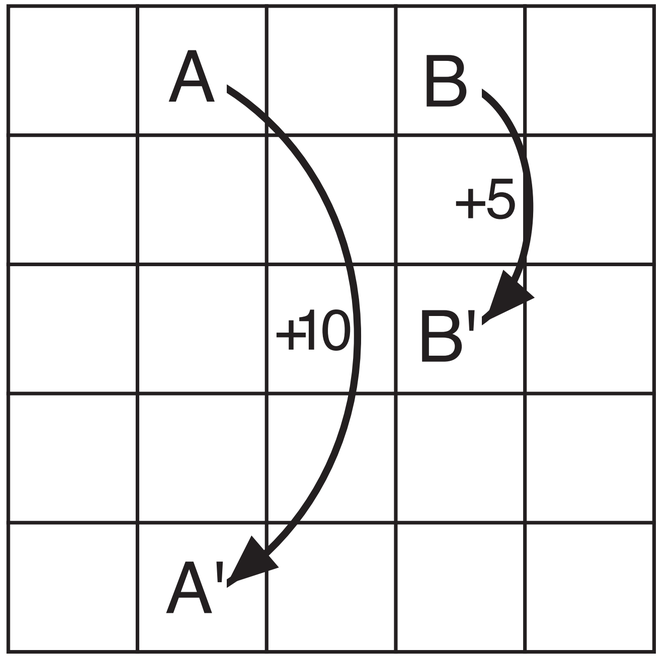


Reward dynamics

1.4	9.7	3.7	5.3	1.0
0.4	2.5	1.8	1.7	0.4
-0.2	0.6	0.6	0.5	-0.1
-0.5	0.0	0.0	0.0	-0.5
-1.0	-0.6	-0.5	-0.5	-1.0

V at k=2

Iterative Policy Evaluation in GridWorld



Reward dynamics

3.4	8.9	4.5	5.3	1.5
1.6	3.0	2.3	1.9	0.6
0.1	0.8	0.7	0.4	-0.4
-1.0	-0.4	-0.3	-0.6	-1.2
-1.9	-1.3	-1.2	-1.4	-2.0

V at k=10,000

Policy Improvement Theorem

Let π and π' be any pair of deterministic policies.

If $\quad q_\pi(s, \pi'(s)) \geq v_\pi(s) \quad \forall s \in \mathcal{S},$

then $\quad v_{\pi'}(s) \geq v_\pi(s) \quad \forall s \in \mathcal{S}\,.$

If you are never worse off at any state by following π' for **one step** and then following π forever after, then following π' **forever** has a higher expected value **at every state**

Policy Improvement Theorem Proof

$$
\begin{aligned}
v_\pi(s) &\leq q_\pi(s, \pi'(s)) \\
&= \mathbb{E}[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s, A_t = \pi'(s)] \\
&= \mathbb{E}_{\pi'}[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s] \\
&\leq \mathbb{E}_{\pi'}[R_{t+1} + \gamma q_\pi(S_{t+1}, \pi'(S_{t+1})) \mid S_t = s] \\
&= \mathbb{E}_{\pi'}[R_{t+1} + \gamma \mathbb{E}_{\pi'}[R_{t+2} + \gamma v_\pi(S_{t+2}) | S_{t+1}, A_{t+1} = \pi'(S_{t+1})] \mid S_t = s] \\
&= \mathbb{E}_{\pi'}\big[R_{t+1} + \gamma R_{t+2} + \gamma^2 v_\pi(S_{t+2}) \;\big|\; S_t = s\big] \\
&\leq \mathbb{E}_{\pi'}\big[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 v_\pi(S_{t+3}) \;\big|\; S_t = s\big] \\
&\vdots \\
&\leq \mathbb{E}_{\pi'}\big[R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 R_{t+4} + \cdots \;\big|\; S_t = s\big] \\
&= v_{\pi'}(s).
\end{aligned}
$$

Greedy Policy Improvement

Given any policy π, we can construct a new greedy policy π' that is guaranteed to be **at least as good**:

$$\begin{aligned}\pi'(s) &\doteq \arg\max_a q_\pi(s,a) \\ &= \arg\max_a \mathbb{E}[R_{t+1} + \gamma v_\pi(S_{T+1}) \,|\, S_t = s, A_t = a] \\ &= \arg\max_a \sum_{s',r} p(s', r \,|\, s, a)\left[r + \gamma v_\pi(s')\right] .\end{aligned}$$

- If this new policy is **not better** than the old policy, then $v_\pi(s) = v_{\pi'}(s)$ for all s (**why?**) Because policy improvement theorem guarantees it is at least as good, so only way for it not to be better is to be the same.
- Also means that the new (and old) policies are **optimal** (**why?**)

If state values are the same after this update, then the Bellman optimality equation is satisfied, and v* is the unique solution to the Bellman optimal

Policy Iteration

$$\pi_0 \xrightarrow{\text{E}} v_{\pi_0} \xrightarrow{\text{I}} \pi_1 \xrightarrow{\text{E}} v_{\pi_1} \xrightarrow{\text{I}} \pi_2 \xrightarrow{\text{E}} \cdots \xrightarrow{\text{I}} \pi_* \xrightarrow{\text{E}} v_*$$

Policy Iteration (using iterative policy evaluation) for estimating $\pi \approx \pi_*$

1. Initialization
   $V(s) \in \mathbb{R}$ and $\pi(s) \in \mathcal{A}(s)$ arbitrarily for all $s \in \mathcal{S}$

2. Policy Evaluation
   Loop:
   $\Delta \leftarrow 0$
   Loop for each $s \in \mathcal{S}$:
   $v \leftarrow V(s)$
   $V(s) \leftarrow \sum_{s',r} p(s', r \mid s, \pi(s)) \big[r + \gamma V(s')\big]$
   $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
   until $\Delta < \theta$ (a small positive number determining the accuracy of estimation)

3. Policy Improvement
   policy-stable $\leftarrow$ *true*
   For each $s \in \mathcal{S}$:
   old-action $\leftarrow \pi(s)$
   $\pi(s) \leftarrow \arg\max_a \sum_{s',r} p(s', r \mid s, a) \big[r + \gamma V(s')\big]$
   If *old-action* $\neq \pi(s)$, then *policy-stable* $\leftarrow$ *false*
   If *policy-stable*, then stop and return $V \approx v_*$ and $\pi \approx \pi_*$; else go to 2

- This is a lot of iterations!
- Is it necessary to run to completion?

Value Iteration

Value iteration interleaves the estimation and improvement steps:

$$v_{k+1}(s) \doteq \max_a \mathbb{E}\left[R_{t+1} + \gamma v_k(S_{t+1}) \,|\, S_t = s, A_t = a\right]$$
$$= \max_a \sum_{s',r} p(s', r \,|\, s, a)\left[r + \gamma v_k(s')\right]$$

Value Iteration, for estimating $\pi \approx \pi_*$

Algorithm parameter: a small threshold $\theta > 0$ determining accuracy of estimation
Initialize $V(s)$, for all $s \in \mathcal{S}^+$, arbitrarily except that $V(terminal) = 0$

Loop:
| $\Delta \leftarrow 0$
| Loop for each $s \in \mathcal{S}$:
| $v \leftarrow V(s)$
| $V(s) \leftarrow \max_a \sum_{s',r} p(s', r \,|\, s, a)\left[r + \gamma V(s')\right]$
| $\Delta \leftarrow \max(\Delta, |v - V(s)|)$
until $\Delta < \theta$

Output a deterministic policy, $\pi \approx \pi_*$, such that
$\pi(s) = \arg\max_a \sum_{s',r} p(s', r \,|\, s, a)\left[r + \gamma V(s')\right]$

Summary

- An **optimal policy** has higher state value than any other policy **at every state**
- A policy's state-value function can be computed by **iterating** an **expected update** based on the Bellman equation
- Given any policy π, we can compute a **greedy improvement** π' by choosing highest expected value action based on v_π
- **Policy iteration:** Repeat:
  Greedy improvement using v_π, then recompute v_π
- **Value iteration:** Repeat:
  Recompute $v\pi$ by assuming greedy improvement at every update