Exact Bayesian Models

CMPUT 366: Intelligent Systems

P&M §10.4

Lecture Outline

1. Recap
2. Model Probabilities
3. Using Model Probabilities
4. Prior Distributions as Bias

Recap: Avoiding Overfitting

There are multiple approaches to avoiding overfitting:

1. **Pseudocounts**: Explicitly account for **regression to the mean**

2. **Regularization**: Explicitly **trade off** between fitting the data and model complexity

3. **Cross-validation**: **Detect** overfitting using some of the training data

Recap: Pseudocounts

- When we have not observed all the **values** of a variable, those variables should not be assigned **probability zero**
- If we don't have very much **data**, we should not be making very extreme predictions
- Solution: artificially add some "pretend" observations for each value of a variable (**pseudocounts**)
    - When there is not much data, predictions will tend to be less extreme (**why**?)
    - When there is more data, the pseudocounts will have less effect on the predictions

Recap: Regularization

- We shouldn't choose a **complicated** model unless there is **clear evidence** for it

- Instead of optimizing directly for training error, optimize training error plus a **penalty** for complexity:

$$\arg\min_{h \in \mathcal{H}} \sum_{e} error(e, h) + \lambda \times regularizer(h)$$

- *regularizer* measures the **complexity** of the hypothesis

- λ is the **regularization parameter**: indicates how important hypothesis complexity is compared to fit

    - Larger λ means complexity is more important

Learning Point Estimates

- So far, we have considered how to find the best **single** model, e.g.,
  - learn ***a*** decision tree
  - optimize ***the*** weights of a linear or logistic regression
- The **predictions** might be a probability distribution, but they are coming out of a single **model**:

$P(Y \mid X)$ Probability of target Y given observation X

- We have been learning **point estimates** of our model

Learning Model Probabilities

- Instead, we could learn a **distribution** over models:
  - $\Pr(X, Y \mid \theta)$ Probability of target Y and features X given model θ
  - $\Pr(\theta \mid D)$ Probability of model θ given dataset D
- This is called **Bayesian learning**: we never discard any model, we only weight them differently depending upon their **posterior probability**
- **Question:** Why would we want to do that?

What is a Model?

- $\Pr(X,Y \mid \theta)$ Probability of target Y and features X given model θ
- $\Pr(\theta \mid D)$ Probability of model θ given dataset D

- We can do Bayesian learning over **finite** sets of models:
  - e.g., { rank by feature $\theta \mid \theta \in$ {height, weight, age} }
- We can do Bayesian learning over **parametric families** of models:
  - e.g., { regression with weights $w_0=\theta_1$, $w_1=\theta_2 \mid \theta \in \mathbb{R}^2$ }
- We can mix the two!
  - θ can encode choice of model family and parameters

What is the Dataset?

- $\Pr(X,Y \mid \theta)$ Probability of target Y and features X given model θ
- $\Pr(\theta \mid D)$ Probability of model θ given dataset D

- We have an expression for the probability of a single example given a model: $\Pr(X, Y \mid \theta)$
- **Question:** What is the expression for the probability of a dataset of observations $D=\{ (X_1,Y_1), \ldots, (X_m,Y_m) \}$ given a model?
  - Easiest approach: Assume that the dataset independent, identically distributed observations: $(X_i,Y_i) \sim P(X, Y \mid \theta)$
  - $$\Pr(D \mid \theta) = \Pr(X_1, Y_1 \mid \theta) \times \ldots \times \Pr(X_m, Y_m \mid \theta)$$
    $$= \prod_{i=1}^{m} \Pr(X_i, Y_i \mid \theta)$$

What is the Posterior Model Probability?

- Pr(X,Y | θ) Probability of target Y and features X given model θ
- Pr(θ | D) Probability of model θ given dataset D

Now we can use **Bayes' Rule** to compute the posterior probability of a model θ:

Prior probability of model θ

$$\Pr(\theta \mid D) = \frac{\boxed{\Pr(D \mid \theta)}\,\boxed{\Pr(\theta)}}{\Pr(D)}$$

$$= \frac{\prod_i \Pr(X_i, Y_i \mid \theta)\Pr(\theta)}{\Pr(D)}$$

$$= \frac{\prod_i \Pr(X_i, Y_i \mid \theta)\Pr(\theta)}{\sum_{\theta'} \Pr(D \mid \theta')\Pr(\theta')}$$

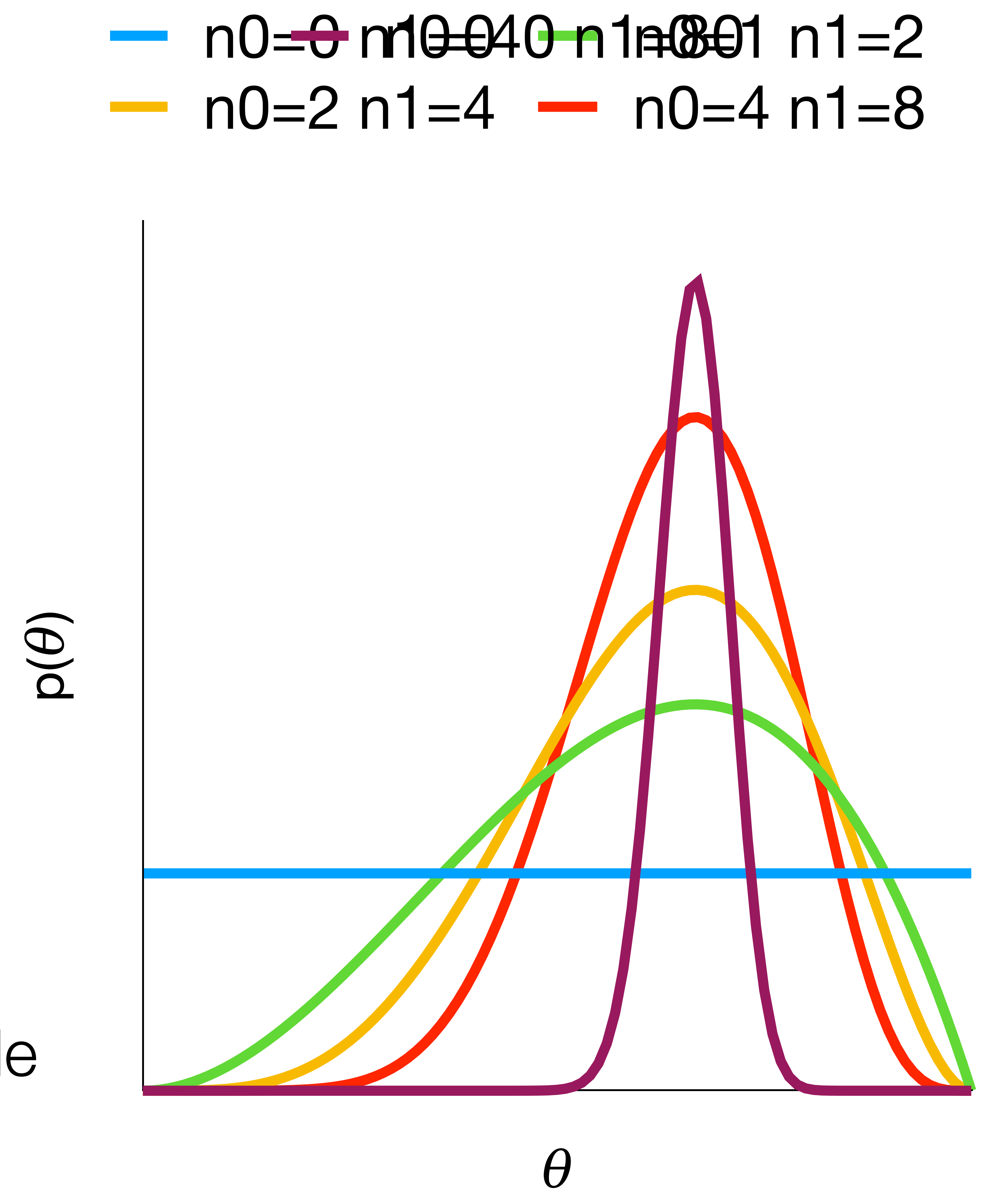
Likelihood of data D given model θ

Example: Biased Coin

- Back to coin flipping! We can flip a coin and observe heads or tails, but we don't know the coin's bias
- Model: **Binomial observations**
  - Observations: $Y \in \{h,t\}$
  - Bias: $\theta \in [0,1]$
  - Likelihood: $\Pr(H \mid \theta) = \theta$
  - **Question:** What should the **prior** $p(\theta)$ be?

Biased Coin: Posterior Probabilities

- Before we see any flips, all biases are equally probable (according to our prior)
- After more and more flips, we become more confident in θ
- θ with **highest probability** is 2/3
  - **Expected** value of θ is less! (**why**?)
  - But with more observations, mode and expected value get **closer**



Beta-Binomial Models

- Likelihood: $P(h \mid \theta) = \theta$
  - aka Bernoulli($h \mid \theta$)
  - Dataset likelihood: $\theta^{n1} \times (1-\theta)^{n0}$
  - aka Binomial(n_1, n_0)
- Prior: $P(\theta) = 1$
  - aka Beta(1,1)
- Models of this kind are called Beta-Binomial models
- They can be solved analytically: $Pr(\theta \mid D) = Beta(1+n_1, 1+n_0)$

Conjugate Priors

- The beta distribution is a **conjgate prior** for the binomial distribution:
    - Updating a beta prior with a binomial likelihood gives a **beta posterior**
- Other distributions have this property:
    - Gaussian-Gaussian (for means)
    - Dirichlet-Multinomial (generalization of Beta-Binomial for multiple values)

Using Model Probabilities

So we can estimate $\Pr(\theta \mid D)$. What can we do with it?

1. Parameter estimates

2. Target predictions (model averaging)

3. Target predictions (point estimates)

1. Parameter Estimates

- Sometimes, we really want to know the parameters of a model itself

- E.g., maybe I don't care about predicting the next coin flip, but I do want to know whether the coin is fair

- Can use $\Pr(\theta \mid D)$ to make statements like $\Pr(0.49 \leq \theta \leq 0.51) > 0.9$

2. Model Averaging

- Sometimes we do want to make predictions:

$$\Pr(Y \mid D) = \sum_{\theta} \Pr(Y \mid \theta) \Pr(\theta \mid D)$$

- This is called the **posterior predictive distribution**

- **Question:** How is this different from just learning a point estimate of a model, and then predicting with that model?

3. Maximum A Posterior

- Sometimes we do want to make predictions, **but...**

$$\Pr(Y|D) = \int_0^1 \Pr(Y|\theta)\Pr(\theta|D)d\theta$$

- the posterior predictive distribution may be **expensive** to compute (or even **intractable**)

- One possible solution is to use the **maximum a posterior** model as a point estimate:

$$\Pr(Y|D) \simeq \Pr(Y|\hat{\theta}) \qquad \text{where } \hat{\theta} = \arg\max_{\theta} \Pr(\theta|D)$$

- **Question:** Why would you do this instead of just using a point estimate that was computed in the usual way?

Prior Distributions as Bias

- Suppose I'm comparing two models, θ_1 and θ_2 such that $\Pr(D \mid \theta_1) = \Pr(D \mid \theta_2)$
- **Question:** Which model has higher **posterior probability**?
- Priors are a way of encoding **bias**: the tell use which models to prefer when the data doesn't

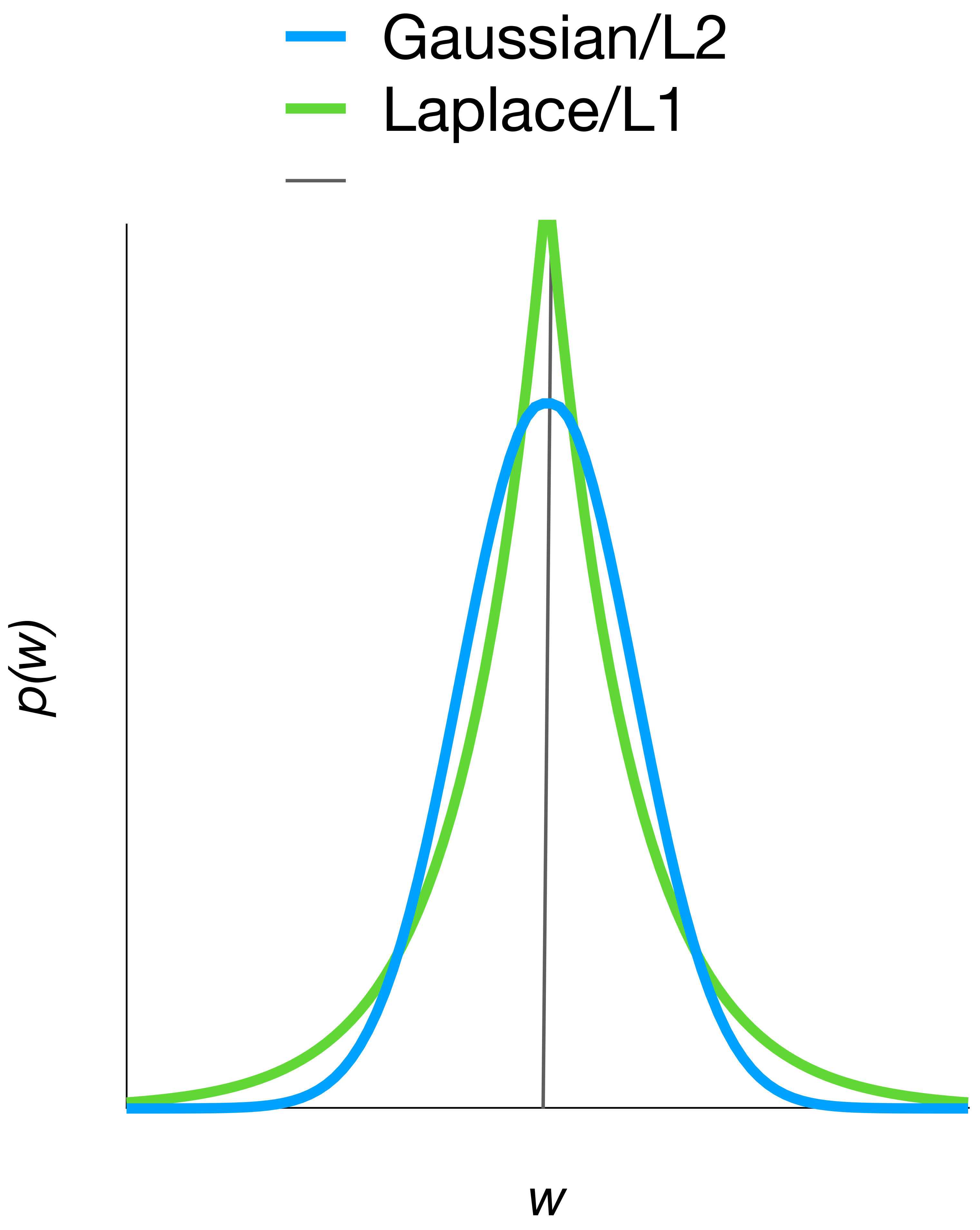
Priors for Pseudocounts

- We can straightforwardly encode pseudocounts as prior information in beta-binomial and dirichlet-multinomial models
- E.g., for pseudocounts k_1 and k_0,

$$p(\theta) = \mathrm{Beta}(1+k_1,\ 1+k_0)$$

Priors for Regularization

- Some **regularizers** can be encoded as priors also
- **L2 regularization** is equivalent to a **Gaussian** prior on the weights: p(w) = N(w|m,s)
- **L1 regularization** is equivalent to a **Laplacian** prior on the weights: p(w) = exp(|w|)/2



Summary

- In Bayesian Learning, we learn a **distribution** over models instead of a **single model**
- When the model is **conjugate**, posterior probabilities can be computed **analytically**
    - See next lecture for non-conjugate models
- We can make predictions by **model averaging** to compute the **posterior predictive distribution**
- The **prior** can encode **bias over models**, much the same as **regularization**
    - In fact, it can exactly encode certain kinds of regularization