Probability Theory

CMPUT 366: Intelligent Systems

P&M §8.1

Recap: Search

- Agent searches **internal representation** to find solution
- **Fully-observable, deterministic, offline, single-agent** problems
- **Graph search** finds a **sequence of actions** to a goal node
  - Efficiency gains from using **heuristic functions** to encode **domain knowledge**
- **Local search** finds a goal node by repeatedly making **small changes** to the current state
  - Random steps and random restarts help handle local optima, completeness

Lecture Outline

1. Recap
2. Uncertainty
3. Probability Semantics
4. Conditional Probability
5. Expected Value

Uncertainty

- In search problems, agent has **perfect knowledge** of the world and its dynamics
- In most applications, an agent cannot just **make assumptions** and then act according to those assumptions
- Knowledge is **uncertain**:
  - Must consider **multiple** hypotheses
  - Must **update** beliefs about which hypotheses are likely given **observations**

Example: Wearing a Seatbelt

- An agent has to decide between **three actions**:
  1. Drive without wearing a seatbelt
  2. Drive while wearing a seatbelt
  3. Stay home
- If the agent thinks that an accident **will** happen, it will just stay home
- If the agent thinks that an accident **will not** happen, it will not bother to wear a seatbelt!
- Wearing a seatbelt only makes sense because the agent is **uncertain** about whether driving will lead to an accident.

Measuring Uncertainty

- **Probability** is a way of **measuring** uncertainty
- We assign a number between 0 and 1 to hypotheses:
  - **0** means absolutely certain that statement is **false**
  - **1** means absolutely certain that statement is **true**
  - **Intermediate** values mean more or less certain
- Probability is a measurement of uncertainty, **not truth**
  - A statement with probability .75 is not "mostly true"
  - Rather, we **believe** it is more **likely** to be true than not

Subjective vs. Objective: The Frequentist Perspective

- Probabilities can be interpreted
  as **objective** statements about the **world**, or
  as **subjective** statements about an agent's **beliefs**

- Objective view is called **frequentist**:

  - The probability of an event is the proportion of times it would happen **in the long run** of repeated experiments

  - Every event has a single, **true** probability

  - Events that can only happen **once** don't have a well-defined probability

Subjective vs. Objective: The Bayesian Perspective

- Probabilities can be interpreted
  as **objective** statements about the **world**, or
  as **subjective** statements about an agent's **beliefs**

- Subjective view is called **Bayesian**:

  - The probability of an event is a measure of an agent's **belief** about its likelihood

  - Different agents can legitimately have **different beliefs**, so they can legitimately assign **different probabilities** to the same event

  - There is only one way to **update** those beliefs in response to new data

- In this course, we will primarily take the **Bayesian** view[

Example: Dice

- Diane rolls a **fair, six-sided die**, and gets the number **X**
  - **Question:** What is **P(X=5)**? (the probability that Diane rolled a 5) A: 1/6
- Diane truthfully tells Oliver that she rolled an **odd** number.
  - **Question:** What should **Oliver** believe P(*X*=5) is? A: 1/3
- Diane truthfully tells Greta that she rolled a number **≥ 5**.
  - **Question:** What should **Greta** believe P(*X*=5) is? A: 1/2
- **Question:** What is **P(X=5)**? A: 1

Semantics: Possible Worlds

- **Random variables** take values from a **domain.**
  We will write them as uppercase letters (e.g., X, Y, D, etc.)

- A **possible world** is a **complete assignment** of values to variables

- A **probability measure** is a function $P : \Omega \to \mathbb{R}$ over **possible worlds** ω satisfying:

  Set of all possible worlds

  1. $$\sum_{\omega \in \Omega} P(\omega) = 1$$
  2. $$P(\omega) \geq 0 \;\; \forall \omega \in \Omega$$

Propositions

- A **primitive proposition** is an equality or inequality expression
  E.g., X = 5 or X ≥ 4

- A **proposition** is built up from other propositions using **logical connectives**.
  E.g., (X=1 ∨ X=3 ∨ X=5)

- The **probability** of a proposition is the sum of the probabilities of the possible worlds in which that proposition is true:

$$P(\alpha) = \sum_{\omega:\omega\models\alpha} P(\omega)$$

$\omega \models \alpha$ means "α is true in ω"

- Therefore:

$$P(\alpha \vee \beta) \geq P(\alpha)$$

$\alpha \vee \beta$ means "α OR β"

$$P(\alpha \wedge \beta) \leq P(\alpha)$$

$\alpha \wedge \beta$ means "α AND β"

$$P(\neg\alpha) = 1 - P(\alpha)$$

$\neg\alpha$ means "NOT α"

Joint Distributions

- In our dice example, there was a **single** random variable
- We typically want to think about the interactions of **multiple** random variables
- A **joint distribution** assigns a probability to each full assignment of values to variables
  - e.g., $P(X=1, Y=5)$. Equivalent to $P(X=1 \wedge Y=5)$
  - Can view this as another way of specifying a single **possible world**

Joint Distribution Example

- What might a day be like in Edmonton? Random variables:
  - **Weather**, with domain {clear, snowing}
  - **Temperature**, with domain {mild, cold, very_cold}
- **Joint distribution** P(Weather, Temperature):

Weather	Temperature	P
clear	mild	0.2
clear	cold	0.3
clear	very cold	0.25
snowing	mild	0.05
snowing	cold	0.15
snowing	very cold	0.1

Marginalization

Question:

What is the **marginal distribution** of Weather?

A: P(clear) = .75
P(snow) = .25

- **Marginalization** is using a joint distribution $P(X_1, \ldots, X_m, \ldots X_n)$ to compute a distribution over a smaller number of variables $P(X_1, \ldots, X_m)$
  - Smaller distribution is called the **marginal distribution** of its variables
- We compute the marginal distribution by summing out the other variables:

$$P(X, Y) = \sum_{z \in dom(Z)} P(X, Y, Z = z)$$

Weather	Temperature	P
clear	mild	0.20
clear	cold	0.30
clear	very cold	0.25
snowing	mild	0.05
snowing	cold	0.10
snowing	very cold	0.10

Conditional Probability

- Agents need to be able to **update** their beliefs based on new **observations**
- This process is called **conditioning**
- We write P(h | e) to denote "probability of **hypothesis** h given that we have observed **evidence** e"
  - P(h | e) is the **probability of h conditional on e**

Semantics of Conditional Probability

- Evidence e lets us **rule out** all of the worlds that are incompatible with e
  - E.g., if I observe that the weather is clear, I should no longer assign **any** probability to the worlds in which it is snowing
  - We need to **normalize** the probabilities of the remaining worlds to ensure that the probabilities of possible worlds sum to 1

$$P(\omega \mid e) \equiv \begin{cases} c \times P(\omega) & \text{if } \omega \models e \\ 0 & \text{otherwise.} \end{cases}$$

$$P(\omega \mid e) \equiv \begin{cases} \frac{1}{P(e)} \times P(\omega) & \text{if } \omega \models e, \\ 0 & \text{otherwise.} \end{cases}$$

Conditional Probability Example

- My initial marginal belief about the weather was:
  P(Weather=snow) = 0.25

- Suppose I observe that the temperature is **mild**.

  - **Question:** What should I **now** believe about the weather?

    A: P(snow) = .05/.25 = .20

1. **Rule out** incompatible worlds

2. **Normalize** remaining probabilities

Weather	Temperature	P
clear	mild	0.20
~~clear~~	~~cold~~	~~0.30~~
~~clear~~	~~very cold~~	~~0.25~~
snowing	mild	0.05
~~snowing~~	~~cold~~	~~0.10~~
~~snowing~~	~~very cold~~	~~0.10~~

Chain Rule

Definition: conditional probability

$$P(h \mid e) = \frac{P(h, e)}{P(e)}$$

- We can run this **in reverse** to get

$$P(h, e) = P(h \mid e) \times P(e)$$

Definition: chain rule

$$\begin{aligned} P(\alpha_1, \ldots, \alpha_n) &= P(\alpha_1) \times P(\alpha_2 \mid \alpha_1) \times \cdots \times P(\alpha_n \mid \alpha_1, \ldots, \alpha_{n-1}) \\ &= \Pi_{i=1}^{n} P(\alpha_i \mid \alpha_1, \ldots, \alpha_{i-1}) \end{aligned}$$

Bayes' Rule

- From the chain rule, we have

$$\begin{aligned} P(h, e) &= P(h \mid e)P(e) \\ &= P(e \mid h)P(h) \end{aligned}$$

- **Often**, P(*e* | *h*) is easier to compute than P(*h* | *e*).

Bayes' Rule:

$$\overset{\text{Posterior}}{\boxed{P(h \mid e)}} = \frac{\overset{\text{Likelihood}}{\boxed{P(e \mid h)}}\ \overset{\text{Prior}}{\boxed{P(h)}}}{\underset{\text{Evidence}}{\boxed{P(e)}}}$$

Expected Value

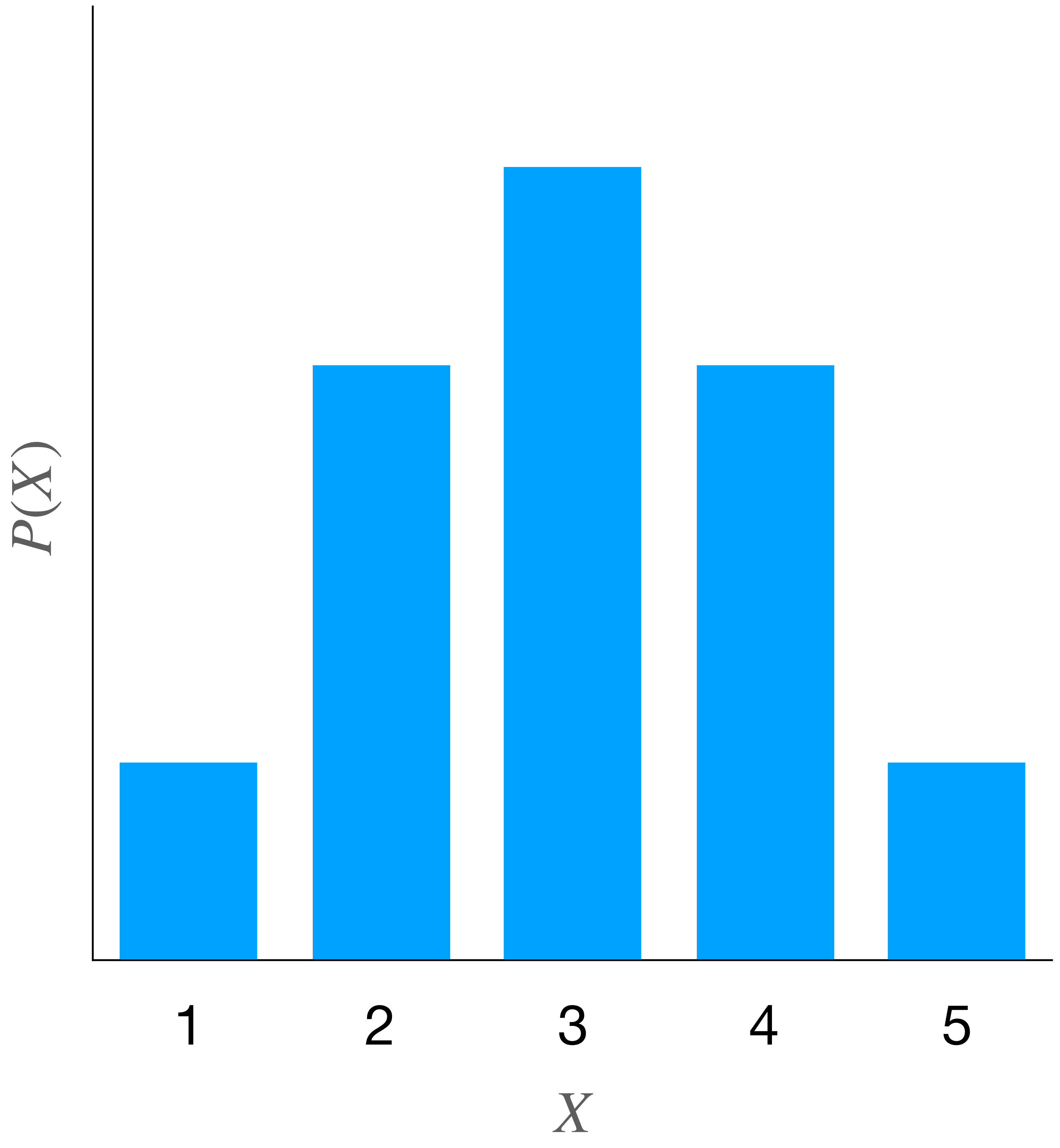
- The **expected value** of a function on a random variable is the weighted **average** of that function over the domain of the random variable, **weighted** by the **probability** of each value:

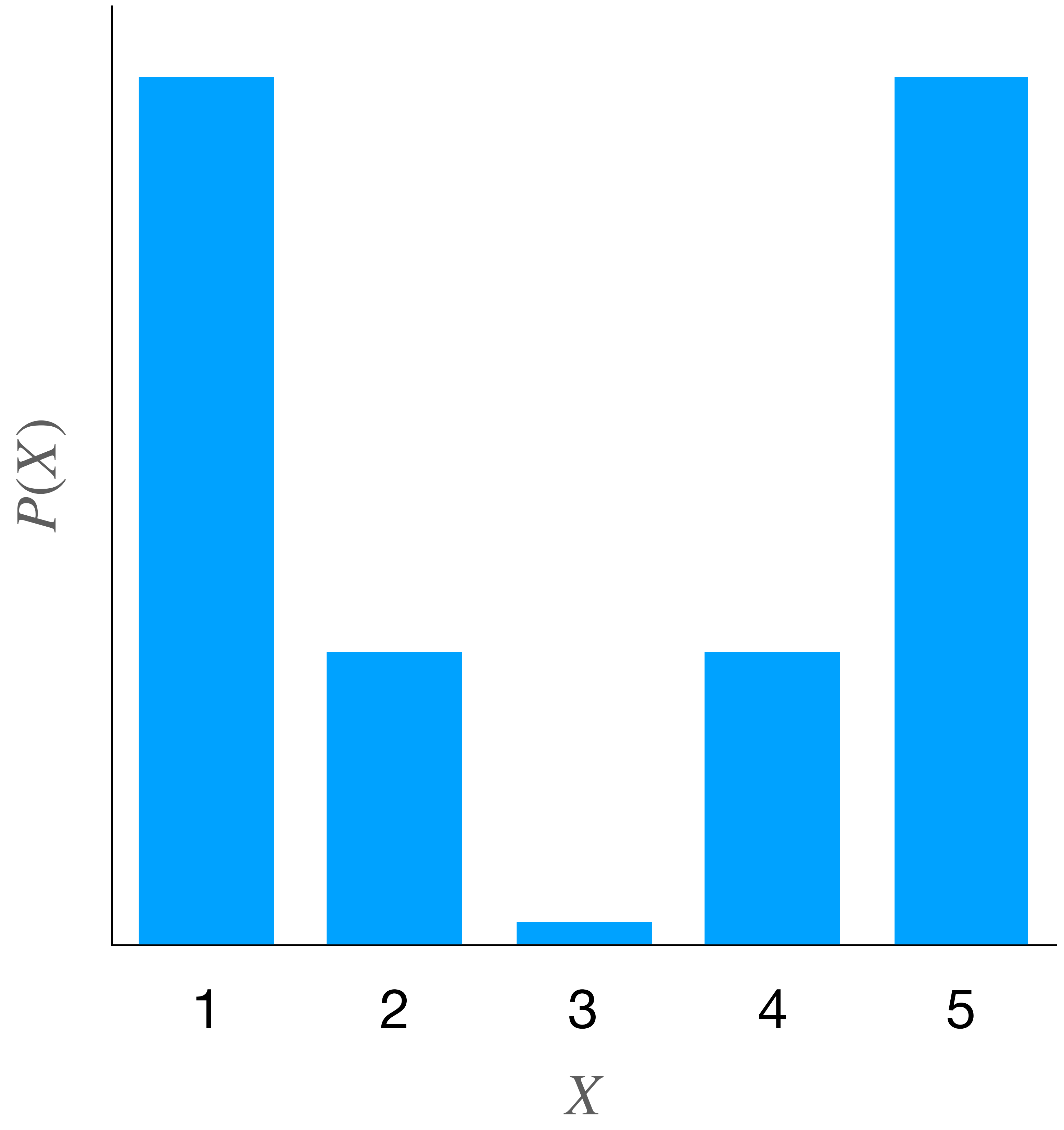
$$\mathbb{E}\left[f(X)\right] = \sum_{x \in dom(X)} P(X = x) f(x)$$

- The **conditional expected value** of a function is the expected value weighted by the **conditional probability**:

$$\mathbb{E}\left[f(X) \mid Y = y\right] = \sum_{x \in dom(X)} P(X = x \mid Y = y) f(x)$$

Expected Value Examples





$$\mathbb{E}[X] = 3$$
$$\mathbb{E}[X^2] \simeq 10$$

$$\mathbb{E}[X] = 3$$
$$\mathbb{E}[X^2] \simeq 12$$

Summary

- Probability is a **numerical** measure of **uncertainty**
- Formal semantics:
  - Weights over **possible worlds** sum to 1
  - Probability of proposition is **total weight** of worlds in which that proposition is true
- **Conditional probability** updates beliefs based on **evidence**
- **Expected value** of a function is its **probability-weighted average** over possible worlds