Branch & Bound

or, How I Learned to Stop Worrying and Love Depth First Search

CMPUT 366: Intelligent Systems

P&M §3.7-3.8

Lecture Outline

1. Recap
2. Cycle Pruning
3. Branch & Bound
4. Exploiting Search Direction

Recap: Heuristics

Definition:
A **heuristic function** is a function $h(n)$ that returns a non-negative estimate of the cost of the cheapest path from n to a goal node.

- e.g., Euclidean distance instead of travelled distance

Definition:
A heuristic function is **admissible** if $h(n)$ is always less than or equal to the cost of the cheapest path from n to a goal node.

- i.e., $h(n)$ is a **lower bound** on cost($\langle n, \ldots, g \rangle$) for any **goal node** g

Recap: A* Search

- A* search uses **both** path cost information and heuristic information to select paths from the frontier
- Let $f(p) = \text{cost}(p) + h(p)$
- $f(p)$ **estimates** the total cost to the nearest goal node **starting from *p***
- A* removes paths from the frontier with **smallest** $f(p)$
- When h is **admissible**,
  $p^* = <s, ..., n, ..., g>$ is a **solution**, and
  $p = <s, ..., n>$ is a **prefix** of p^*:
    - $f(p) \leq \text{cost}(p^*)$

$$\underbrace{\underbrace{\text{start} \xrightarrow{\text{actual}}}_{\text{cost(p)}} \; n \; \underbrace{\xrightarrow{\text{estimated}} \text{goal}}_{\text{h(n)}}}_{\text{f(p)}}$$

Cycle Pruning

- Even on **finite graphs**, depth-first search may not be complete, because it can get trapped in a **cycle**.
- A search algorithm can **prune** any path that ends in a node already on the path **without missing an optimal solution**
  (**Why?**)

Questions:

1. Is depth-first search on with cycle pruning **complete** for finite graphs?
2. What is the time complexity for cycle checking in **depth-first search**?
3. What is the time complexity for cycle checking in **breadth-first search**?

Cycle Pruning
Depth First Search

Input: a *graph*; a set of *start nodes*; a *goal* function

frontier := { $<s>$ | s is a start node}
while *frontier* is not empty:
    select the newest **path** $<n_1, n_2, \ldots, n_k>$ **from** ***frontier***
    remove $<n_1, n_2, \ldots, n_k>$ from *frontier*
    if $n_k \neq n_j$ for all $1 \leq j < k$:
        if *goal*(n_k):
            return $<n_1, n_2, \ldots, n_k>$
        for each neighbour n of n_k:
            add $<n_1, n_2, \ldots, n_k, n>$ to *frontier*
end while

Heuristic Depth First Search

	Heuristic Depth First	A*	Branch & Bound
Space complexity	$O(mb)$	$O(b^m)$	$O(mb)$
Heuristic Usage	Limited	Optimal	Optimal (if bound low enough)
Optimal?	No	Yes	Yes (if bound high enough)

Branch & Bound

- The *f*(*p*) function provides a **path-specific lower bound** on solution cost starting from *p*
- **Idea:** Maintain a **global upper bound** on solution cost also
  - Then prune any path whose lower bound **exceeds** the upper bound
- **Question:** Where does the upper bound come from?
  - **Cheapest** solution found so far
  - Before solutions found, specified on entry
  - Can increase the global upper bound **iteratively** (as in iterative deepening search)

Branch & Bound Algorithm

Input: a *graph*; a set of *start nodes*; a *goal* function; heuristic $h(n)$; $bound_0$

frontier := { $\langle s \rangle$ | *s* is a start node}
$bound := bound_0$
$best := \varnothing$
while *frontier* is not empty:
    select the newest path $\langle n_1, n_2, \dots, n_k \rangle$ from *frontier*
    remove $\langle n_1, n_2, \dots, n_k \rangle$ from *frontier*
    if $\mathrm{cost}(\langle n_1, n_2, \dots, n_k \rangle) + h(n_k) \leq bound$:
        if $goal(n_k)$:
            $bound := \mathrm{cost}(\langle n_1, n_2, \dots, n_k \rangle)$
            $best := \langle n_1, n_2, \dots, n_k \rangle$
        else:
            for each neighbour *n* of n_k:
                add $\langle n_1, n_2, \dots, n_k, n \rangle$ to *frontier*
end while
return *best*

Branch & Bound Analysis

- If $bound_0$ is set to just above the optimal cost, branch & bound will explore no more paths than A*
(**Why?**)

- With **iterative increasing** of $bound_0$, will re-explore some lower-cost paths, but still similar time-complexity to A*
Question: How *much* should the bound get increased by?

    - Iteratively increase bound to the **lowest-f-value** node that was **pruned**

    - Worse than A* by no more than a **linear** factor of *m*,
by the same argument as for iterative deepening search

Exploiting Search Direction

- When we care about finding the path to a known goal node, we can search forward, but we can often search **backward**

- Given a search graph $G=(N,A)$, **known** goal node g, and set of start nodes S, can construct a **reverse search problem** $G=(N, A^r)$:

    1. Designate g as the start node

    2. $A^r = \{ <n_2,n_1> \mid <n_1,n_2> \in A \}$

    3. $\text{goal}^r(n) = \text{True if } n \in S$
       (i.e., if n is a start node of the original problem)

Questions:

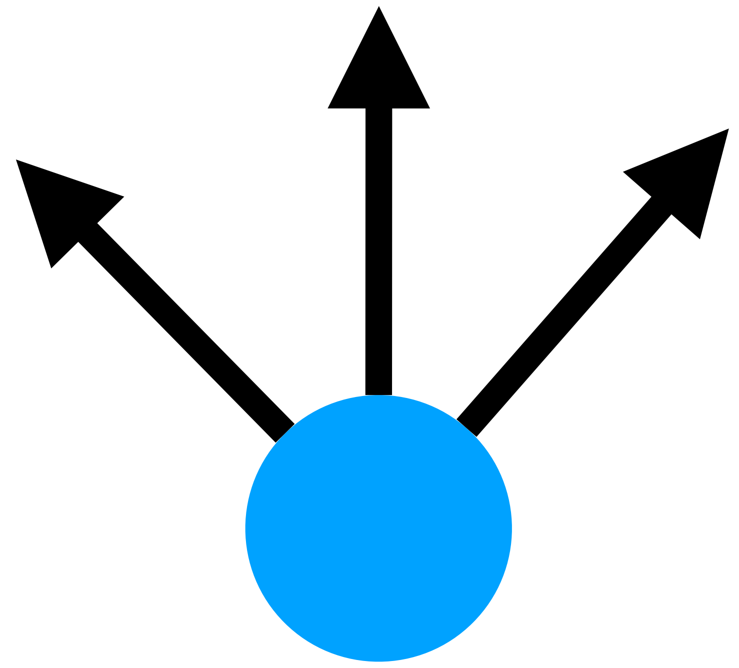
1. When is this **useful**?

2. When is this **infeasible**?

Reverse Search

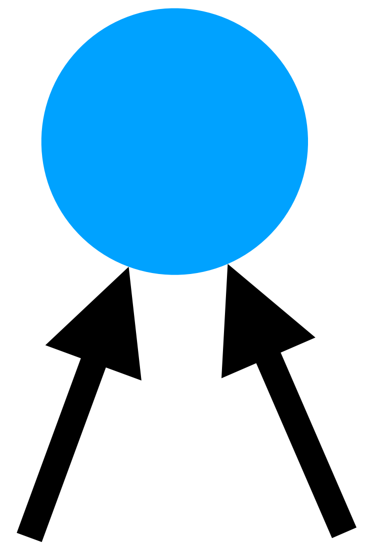
Definitions:

1. **Forward branch factor**: Maximum number of **outgoing** neighbours
   Notation: *b*

   - Time complexity of forward search: $O(b^m)$

2. **Reverse branch factor**: Maximum number of **incoming** neighbours
   Notation: r

   - Time complexity of reverse search: $O(r^m)$

When the reverse branch factor is **smaller** than the forward branch factor, reverse search is more **time-efficient**.

Bidirectional Search

- **Idea:** Search backward from from goal and forward from start **simultaneously**
- Time complexity is **exponential in path length**, so exploring half the path length is an exponential improvement
  - Even though must explore half the path length **twice**
- Main problems:
  - **Ensuring** that the frontiers meet
  - **Checking** that the frontiers have met

Questions:

What bidirectional **combinations** of search algorithm make sense?

- Breadth first + Breadth first?
- Depth first + Depth first?
- Breadth first + Depth first?

Summary

- **Cycle pruning** can guarantee the **completeness** of depth-first search on **finite** graphs
  - Although depth first search is really most useful on very large or **infinite** graphs...
- **Branch & bound** combines the **optimality** guarantee and **heuristic efficiency** of A* with the space efficiency of depth-first search
- Tweaking the **direction of search** can yield efficiency gains